Effective Kan fibrations in simplicial sets

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Preliminary remarks

- Contents are based on arXiv:2009.12670
- Joint work with Eric Faber
- Eric will give an online talk on the same topic at the Moravian Café (aka Masaryk University Algebra Seminar) on March 11

Question

Question

What is the correct constructive definition of a Kan fibration between simplicial sets?

Motivation comes from homotopy type theory.

I believe our work is also of interest to classical homotopy theorists: in fact, the maths is in our paper is (for the most part) new classical maths as well.

Section 1

Type theory

Constructive type theory

Martin-Löf's constructive type theory is:

- a foundation for constructive mathematics.
- a functional programming language.
- basis for proof assistants such as Agda and Lean.

Proofs as programs

$$\Gamma \vdash a \in \sigma$$

has two readings:

- a is a proof of the proposition σ (in context Γ).
- a is a computer program meeting the specification σ (in context Γ).

Important properties

- *Decidability of type checking:* The question whether

$$\Gamma \vdash a \in \sigma$$

is derivable or not is decidable.

- *Normalisation:* If $\Gamma \vdash a \in \sigma$ is derivable, then there is a term t is normal form and $\Gamma \vdash a = t \in \sigma$ is derivable as well.
- *Canonicity:* Because the only closed terms of type $\mathbb{N}$ in normal form are numerals $S^n 0$, we have that for each term t for which $\vdash t \in \mathbb{N}$ is derivable, there is a numeral $S^n 0$ for which $\vdash t = S^n 0 \in \mathbb{N}$ is derivable as well.

Issues

- Lack of function extensionality
- No quotient types
- No “bracketing”

Homotopy type theory

Slogan

Types are homotopy types (of spaces).

This interpretation justifies new proof principles:

- Univalence
- Higher inductive types

We get:

- Function extensionality
- Quotient types
- Propositional truncation

as consequences.

Problem

Adding axioms to type theory may destroy normalisation and canonicity.

Cubical type theory

A type theory

- in univalence and (some) higher inductive types can be derived.
- which enjoys canonicity (Huber).
- which enjoys homotopy canonicity (Coquand, Huber, Sattler).
- which enjoys normalisation (Sterling, Angiuli).

Key step: Coquand's definition of a uniform Kan fibration in cubical sets.
(These interpret the dependent types.)

Simplicial type theory?

Could there be a similar type theory based on simplicial sets?

Some advantages:

- Types will be interpreted as homotopy types
- There is only one variant of simplicial sets
- Simplicial techniques are pervasive in homotopy theory

Key step should again be: find the correct constructive definition of a Kan fibration of simplicial sets.

Section 2

Effective Kan fibrations

Our contribution

Our paper

In our paper we define a notion of an effective Kan fibration in simplicial sets such that ...

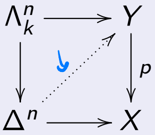
- they are closed under Π, constructively.
- effective Kan fibration have the RLP with respect to horn inclusions, constructively.
- every map which has the RLP with respect to horn inclusions can be equipped with the structure of a effective Kan fibration, classically.
- one can, constructively, obtain universal effective Kan fibrations.

Our definition is the first to satisfy all these properties.

Kan fibrations

Definition

A map $p : Y \to X$ is a *Kan fibration* if for any horn $\Lambda^n_k \to \Delta^n$ and any commutative diagram



there exists a dotted arrow making both triangles commute.

First question

Should we only demand the existence of such fillers (property) or should we say that a Kan fibration is a map equipped with a choice of fillers (structure)?

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$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & X \end{array}$$

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First question

Should we only demand the existence of such fillers (property) or should we say that a Kan fibration is a map equipped with a choice of fillers (structure)?

Our answer

It should be structure!

Uniformity conditions

Definition

A map $p : Y \to X$ is a *algebraic Kan fibration* if for any commutative diagram of the form

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \longrightarrow & X \end{array}$$

it comes equipped with a choice of filler (the dotted arrow).

Second question

Should these fillers satisfy some compatibility conditions?

Uniformity conditions

Definition

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Second question

Should these fillers satisfy some compatibility conditions?

Our answer

Some compatibility ("uniformity") conditions should be satisfied!

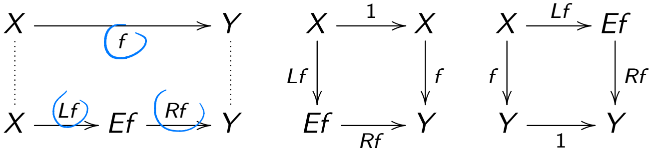
But which ones? To state these, we use the language of algebraic weak factorisation systems.

Algebraic weak factorisation systems

Functorial factorisation

A functor $\mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$ is a *functorial factorisation* on a category $\mathcal{C}$ if it is a section of the composition functor $\circ : \mathcal{C}^{\to\to} \to \mathcal{C}^{\to}$.

So a functorial factorisation writes every map f in $\mathcal{C}$ as a composition:



This turns the functors L and R into (co)pointed endofunctors $\mathcal{C}^{\to} \to \mathcal{C}^{\to}$.

Algebraic weak factorisation system (Grandis-Tholen,Garner)

A functorial factorisation is an algebraic weak factorisation system (AWFS) if L and R can be extended to a comonad and a monad on $\mathcal{C}^{\to}$, respectively, and a distributive law holds (for the comonad over the monad).

Left and right maps

Given an AWFS:

- a *left map* is a coalgebra for the comonad.
- a *right map* is an algebra for the monad.

Both are closed under composition and the left maps have the LLP wrt to the right maps.

Due to the distributive law both classes determine each other.

But the classes are *not* closed under retracts. Their retract closures give one an ordinary weak factorisation system.

See Bourke and Garner, *Algebraic weak factorisation systems I: Accessible AWFS*.

Cofibrations

Our definition of an effective Kan fibration relies on the presence of two AWFSs on the category of simplicial sets.

Cofibrations, constructively

A map $f : Y \to X$ in simplicial sets is a *cofibration* if it is a monomorphism, and given any $x \in X_n$, we can decide whether x lies in the image of f, and if so, we can effectively find the $y \in Y_n$ such that $f_n(y) = x$.

These cofibrations form the left class in an AFWS. The associated right class we will call the *effective trivial Kan fibrations*.

Simplicial Moore path object

In Van den Berg & Garner, we defined a simplicial Moore path functor. The idea is that there is an endofunctor M on simplicial sets together with natural transformations $r : X \to MX$, $s, t : MX \to X$ and $\circ : MX \times_X MX \to MX$ equipping X with the structure of an internal category. In addition, there is a contraction $\Gamma : MX \to MMX$.

This gives one a notion of M-homotopy, which defines a congruence on the category of simplicial sets. Homotopic maps (wrt to $\mathbb{I} = \Delta^1$) are also M-homotopic, and for maps between Kan complexes the converse holds as well.

Theorem

The functorial factorisation sending $f : Y \to X$ to

$$Y \xrightarrow{(r.f,1)} MX \times_X Y \xrightarrow{s.p_1} X$$

is part of an algebraic weak factorisation system.

Naive fibrations

Naive fibrations

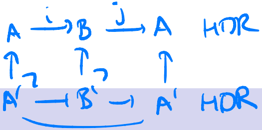
A *naive fibration* is a right map for this AWFS: that is, a map $p : Y \to X$ which comes equipped with a transport operation

$$T : MX \times_X Y \to Y$$

with $p.T = s.p_1,\ T.(r.p, 1) = 1$ and
$T.(\mu.(p_1, p_2), p_3) = T.(p_1, T.(p_2, p_3))$.

Kan fibrations are naive fibrations, but the converse is false. Indeed, every map $X \to 1$ is a naive fibration.

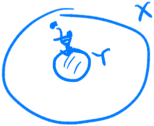
Hyperdeformation retracts



Hyperdeformation retracts (HDRs)

A *hyperdeformation retract* is a left map for this AWFS: that is, a map $i : Y \to X$ for which there is a retraction $j : X \to Y$ and an M-homotopy $H : X \to MX$ with $H : 1 \simeq i.j$ such that $\Gamma.H = PH.H$.

Important example: $d_i / d_{i+1} : \Delta^n \to \Delta^{n+1}$ with s_i as retraction.



Facts:

- The functor
$$\mathrm{dom} : \mathrm{HDR} \to \mathrm{sSets}$$
is a Grothendieck fibration. (It is actually a bifibration, but we won't need that today.)
- The category of HDRs has pullbacks, and cartesian morphisms are stable under pullback.

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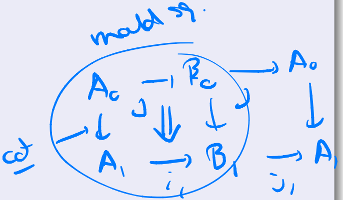
Mould square

Definition

A *mould square* is a morphism of HDRs which is cartesian over a cofibration. That is, it is a square of the form

$$\begin{array}{ccc} A_0 & \xrightarrow{i_0} & B_0 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_1 & \xrightarrow[i_1]{} & B_1 \end{array}$$

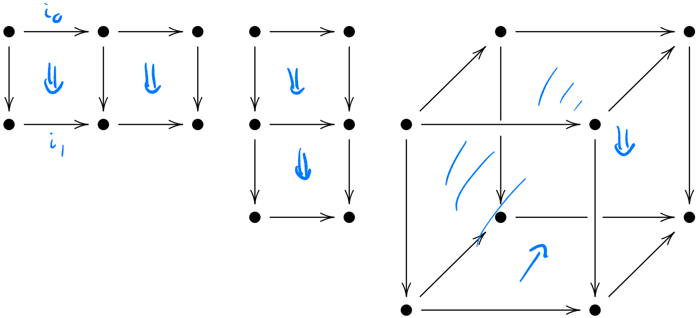


in which i_0 and i_1 are HDRs and the square is a cartesian morphism of HDRs in which a and b are cofibrations.

Properties of mould squares

Properties of mould squares

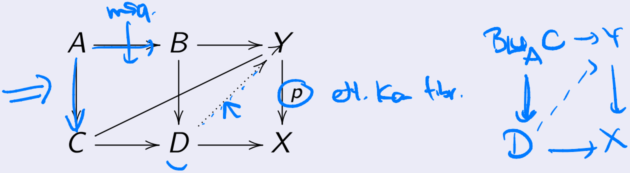
Mould squares can be composed horizontally and vertically, and they can be pulled back along arbitrary morphisms of HDRs (base change).



Effective Kan fibration

Definition

To equip a map $p : Y \to X$ with the structure of an *effective Kan fibration* means that one should specify for any solid commutative diagram



a morphism $D \to Y$ making everything commute, in a way which respects horizontal and vertical composition, as well as base change of mould squares.

Note that the lifting property (without the compatibility conditions) just says that the map p has the RLP against the induced map from the pushout $B \cup_A C$ to D.

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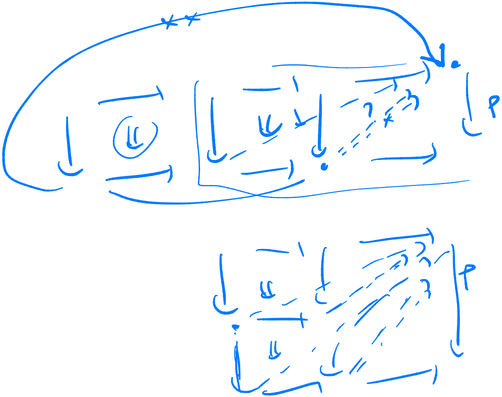

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Horn squares

Proposition

Effective Kan fibration have the RLP wrt Horn inclusions.

Proof.

There is a special class of mould squares, which we call *horn squares*:

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & s_i^*(\partial\Delta^n) & \longrightarrow & \partial\Delta^n \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & \xrightarrow[d_i/d_{i+1}]{} & \Delta^{n+1} & \xrightarrow[s_i]{} & \Delta^n \end{array}$$

The induced map from the pushout to the bottom-right object of the left hand square is the horn inclusion $\Lambda^{n+1}_{i/i+1} \to \Delta^{n+1}$. Therefore effective Kan fibration have the RLP wrt horn inclusions. □

Classically OK

In fact, one can show (with quite some effort!) that the compatibility conditions ensure that the lifts against the horn squares determine the lifts against all the mould squares. As a consequence, the compatibility conditions can be expressed purely as conditions on the lifts against horn squares. This can be used to show:

Theorem

Classically (in **ZFC**) every Kan fibration can be equipped with the structure of an effective Kan fibration.

Theorem

Universal effective Kan fibrations exist (also constructively).

Gambino & Sattler

Gambino and Sattler have also proposed a definition of a uniform Kan fibration (mimicking Coquand's definition in cubical sets).

Proposition

Effective Kan fibration are uniform Kan fibration in the sense of Gambino & Sattler. (I expect the converse to be false (constructively!).)

Proof.

If $m : A \to B$ is a cofibration, then we obtain a mould square on the left in:

$$\begin{array}{ccccc} A & \xrightarrow{A \times \delta_i} & A \times \mathbb{I} & \xrightarrow{\pi_A} & A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m \times \mathbb{I}} & & \downarrow{\scriptstyle m} \\ B & \xrightarrow[B \times \delta_i]{} & B \times \mathbb{I} & \xrightarrow[\pi_B]{} & B \end{array}$$

Therefore effective Kan fibration have the RLP against maps of the form $m \hat{\otimes} \delta_i$ with $\delta_i : 1 \to \mathbb{I}$ being one of the endpoint inclusions. Their uniformity conditions follow from our base change condition. □

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Towards an algebraic model structure

We also have proofs of:

- the existence of a model structure on the simplicial sets, when restricted to those that are effectively Kan.
- the existence of a model of type theory with $\Pi, \Sigma, \mathbb{N}, 0, 1, +, \times$.

We are currently working on:

- the existence of an algebraic model structure on simplicial sets.
- the existence of a model of univalent type theory in simplicial sets.

Future work

What remains to be proven (constructively!):

- We can show that universal effective Kan fibration exist, but we haven't shown they are univalent.
- We haven't shown that universes are effectivily Kan.
- And we haven't shown that there exists an algebraic model structure on the entire category of simplicial sets based on our notion of an effective Kan fibration.

THANK YOU!

First definition of M

Let $\mathbb{T}_0$ be the simplicial set whose n-simplices are zigzags (*traversals*) of the form

$$\bullet \xleftarrow{p_1} \bullet \xrightarrow{p_2} \bullet \xrightarrow{p_3} \bullet \xleftarrow{p_4} \bullet \xrightarrow{p_5} \bullet$$

with $p_i \in [n]$. With the final segment ordering this can be seen as a poset internal to simplicial sets (simplicial poset). Then M can be defined as the polynomial functor associated to the map $\mathrm{cod} : \mathbb{T}_1 \to \mathbb{T}_0$.

This makes M an instance of a *polynomial comonad*.

Second definition of M

Alternatively, we can define for any such n-dimensional traversal θ its *geometric realisation* $\widehat{\theta}$ (often just written θ) as the colimit of the diagram

$$\begin{array}{ccccccccccc} \Delta^n & & & & \Delta^n & & & & \Delta^n & & \cdots & & & & \Delta^n \\ & \searrow^{d_{p_1^s}} & & \swarrow_{d_{p_1^t}} & & \searrow^{d_{p_2^s}} & & \swarrow_{d_{p_2^t}} & & \searrow^{d_{p_3^s}} & & \swarrow_{d_{p_3^t}} \quad \searrow^{d_{p_k^s}} & & \swarrow_{d_{p_k^t}} & \\ & & \Delta^{n+1} & & & & \Delta^{n+1} & & & & \Delta^{n+1} & & \Delta^{n+1} & & \end{array}$$

where p_i^s is $p_i + 1$ if edge i point to the right, and p_i^s is p_i if edge i points to the left and *vice versa* for p_i^t. Then

$$(MX)_n = \sum_{\theta \in (\mathbb{T}_0)_n} \mathrm{Hom}(\widehat{\theta}, X).$$

By considering only those n-dimensional traversals of the form

$$\bullet \xrightarrow{n} \bullet \xrightarrow{n-1} \dots \xrightarrow{1} \bullet \xrightarrow{0} \bullet$$

one can show that $X^{\mathbb{I}} \subseteq MX$.