The Strategic Balance of Games in Logic

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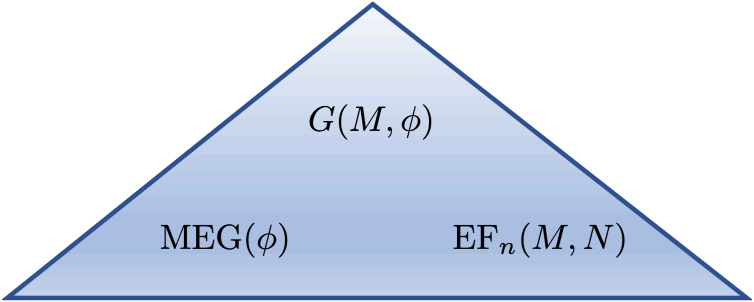
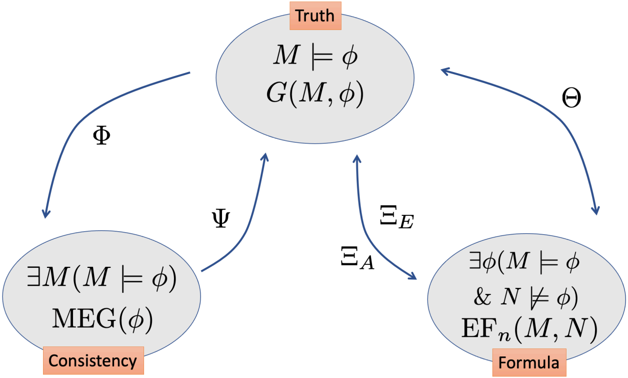
- "The chess-board is the world, the pieces are the phenomena of the universe, the rules of the game are what we call the laws of Nature. The player on the other side is hidden from us." (Thomas Huxley)

1. Evaluation Game: "ϕ is true in M?"
2. Model Existence Game: "ϕ is consistent?"
3. EF (Ehrenfeucht-Fraïssé) game: "some sentence is true in M but false in N?"

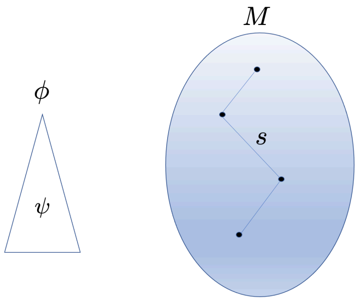
Really just one game. Essential to logic. Distinguishes logic from algebra, topology, analysis, etc.


$G(M, \phi)$
$\mathrm{MEG}(\phi)$
$\mathrm{EF}_n(M, N)$


Truth
$M \models \phi$
$G(M, \phi)$
Φ
Ψ
Θ
Ξ_E
Ξ_A
$\exists M(M \models \phi)$
$\mathrm{MEG}(\phi)$
Consistency
$\exists \phi(M \models \phi$
$\&\ N \not\models \phi)$
$\mathrm{EF}_n(M, N)$
Formula

Evaluation (a.k.a. semantic) Game $G(M, \phi)$

- Two players Abelard and Eloise.
- M a model, ϕ a sentence.
- s an assignment.
- Pairs (ψ, s) are positions.
- Starting position is $(\phi, \emptyset)$.



Evaluation (a.k.a. semantic) Game $G(M, \phi)$

- Suppose s is an assignment. $\mathrm{Diag}_M(s)$ = the set of all literals i.e. atomic and negated atomic formulas that s satisfies in M.
- $\neg, \wedge, \vee, \forall, \exists$.
- Negation Normal Form (for simplicity!).
- Intuitively, Eloise defends the proposition that ϕ is (informally) true in M and Abelard doubts it.

The **rules** in position (ψ, s) are:

(1) If ψ is a literal, the game ends and Eloise wins if $\psi \in \mathrm{Diag}_M(s)$. Otherwise Abelard wins.

(2) If ψ is $\psi_0 \wedge \psi_1$, then Abelard chooses whether the next position is (ψ_0, s) or (ψ_1, s).

(3) If ψ is $\psi_0 \vee \psi_1$, then Eloise chooses whether the next position is (ψ_0, s) or (ψ_1, s).

(4) If ψ is $\forall x \theta$, then Abelard chooses $a \in M$ and the next position is $(\theta, s(a/x))$.

(5) If ψ is $\exists x \theta$, then Eloise chooses $a \in M$ and the next position is $(\theta, s(a/x))$.

- We say that ϕ is true in M if Eloise has a winning strategy in $G(M, \phi)$.
- This is the game-theoretical meaning of truth in a model.
- We can go further and say that the *game* $G(M, \phi)$ is the meaning of ϕ in M. Here meaning would be a broader concept than the mere truth or falsity of ϕ.
- [Wittgenstein, 1953], [Henkin, 1961], [Hintikka, 1968]

- The game $G(M, \phi)$ reflects the syntactical structure of ϕ.
- The game $G(M, \phi \wedge \psi)$ is intimately related to the two games $G(M, \phi)$ and $G(M, \psi)$.
- The same with $G(M, \phi \vee \psi)$, $G(M, \exists x\phi)$ and $G(M, \forall x\phi)$.
- This phenomenon is a manifestation of the broader concept of *compositionality*.
- The games $G(M \times N, \phi)$, $G(M + N, \phi)$, and $G(\Pi_i M_i/F, \phi)$ are intimately related to the games $G(M, \phi)$, $G(N, \phi)$ and $G(M_i, \phi)$ [Feferman, 1972].

- If ϕ is **propositional** i.e. has only zero-place relation symbols and no constant or function symbols, and no quantifiers, then only moves (1)-(3) occur in $G(M, \phi)$, and the assignments can be forgotten.
- If ϕ is **universal**, the game $G(M, \phi)$ has no moves of type (5).
- If it is **existential**, the game has no moves of type (4).
- If **universal-existential**, then all type (5) moves come before type (4) moves.
- If we add **new logical operations** to our logic, such as infinite conjunctions and disjunctions, generalized quantifiers or higher order quantifiers, it is clear how to modify the game $G(M, \phi)$ to accommodate the new logical operations.

For example, for ϕ in $L_{\infty\omega}$, we modify above (2) and (3) as follows:

(2') If ψ is $\bigwedge_{i\in I}\psi_i$, then Abelard chooses $i \in I$ and the next position is (ψ_i, s).

(3') If ψ is $\bigvee_{i\in I}\psi_i$, then Eloise chooses $i \in I$ and the next position is (ψ_i, s).

Similarly for generalized quantifiers.

Modal logic

Finally, if M is a Kripke-model and ϕ a sentence of modal logic, the game $G(M, \phi)$ is entirely similar. The assignments have a singleton domain $\{x_0\}$ and values in the frame of M. The moves corresponding to $\diamond$ and $\Box$ are like (4) and (5):

(4') If ψ is $\Box\theta$, then Abelard chooses a node b accessible from $s(x_0)$ and the next position is $(\theta, s(b/x_0))$.

(5') If ψ is $\diamond\theta$, then Eloise chooses a node b accessible from $s(x_0)$ and the next position is $(\theta, s(b/x_0))$.

- The game $G(M, \phi)$ is useful in finding a **countable** submodel N of M with desired properties.
- For any strategy τ of Eloise in $G(M, \phi)$ let $T(M, \tau)$ be the set of countable submodels N of M such that N is closed under τ i.e. if Abelard plays in (4) always $a \in N$, then also Eloise plays in (5) always $b \in N$.
- Note that if $N \in T(M, \tau)$, then τ is a strategy of Eloise also in $G(N, \phi)$. Moreover, if τ is a winning strategy in $G(M, \phi)$, then it is also a winning strategy in $G(N, \phi)$.
- The **Löwenheim-Skolem Theorem** of $L_{\omega_1\omega}$ is essentially the statement that $T(M, \phi) \neq \emptyset$, when $\phi \in L_{\omega_1\omega}$.

In conclusion, the game $G(M, \phi)$ is a versatile tool for understanding the meaning of a logical sentence ϕ in a mathematical structure M, or even in V.

Model Existence Game MEG(ϕ)

- We have a sentence and we ask whether the sentence has a model. Thus this is about *consistency* and its opposite, *contradiction*.
- Is there some model M such that Eloise can win $G(M, \phi)$?
- Suppose ϕ is a first order sentence. Logical operations: $\neg, \wedge, \vee, \forall$ and $\exists$.
- We assume ϕ is in Negation Normal Form.

- The game MEG(ϕ) has two players Abelard and Eloise.
- Intuitively, Eloise defends the proposition that ϕ has a model and Abelard doubts it. Abelard expresses his doubt by asking questions.
- We let $C = \{c_0, c_1, \ldots, c_n, \ldots\}$ be a set of new constant symbols. Intuitively these are names of elements of the supposed model.

Model Existence Game

A position is a finite set S of pairs (ψ, s), where s is an assignment into C. Starting position is $\{(\phi, \emptyset)\}$. Abelard chooses a pair $(\psi, s) \in S$.

(1) $(\psi_0 \wedge \psi_1, s)$: Next position is $S \cup \{(\psi_0, s)\}$ or $S \cup \{(\psi_1, s)\}$ (Abelard decides which).

(2) If $(\psi_0 \vee \psi_1, s)$: Next position is $S \cup \{(\psi_0, s)\}$ or $S \cup \{(\psi_1, s)\}$ (Eloise decides which).

(3) If $(\forall x\theta, s)$: Next position is $S \cup \{(\theta, s(c/x))\}$ (Abelard chooses $c \in C$).

(4) If $(\exists x\theta, s)$: Next position is $S \cup \{(\theta, s(c/x))\}$ (Eloise chooses $c \in C$).

If $(\psi, s), (\neg\psi, s') \in S$, where $s(x) = s'(x)$ for all free x in ψ, Abelard wins.

- Gentzen's natural deduction,
- [Beth, 1955],
- [Hintikka, 1955],
- [Smullyan, 1963],
- [Makkai, 1969].
- Craig Interpolation Theorem.
- Completeness Theorem.
- Preservations Theorems.

Truth $\Rightarrow$ consistency

Theorem

Every strategy τ of Eloise in $\boldsymbol{G(M, \phi)}$ determines a strategy $\Phi(\tau)$ of Eloise in $\mathrm{MEG}(\phi)$. If τ is a winning strategy, then so is $\Phi(\tau)$.

(We assume the vocabulary of M is countable.)

- There is a countable submodel N of M such that τ is a strategy of Eloise in $G(N, \phi)$. Let $\pi : C \to N$ be an onto map.
- A pair (ψ, s) is a τ-position if there is there is some sequence of positions in $G(N, \phi)$, following the rules of $G(N, \phi)$ starting with $(\phi, \emptyset)$, Eloise using τ, which ends at (ψ, s).
- A C-translation of the τ-position (ψ, s) is a pair (ψ, s') where s' is a C-assignment with $\pi(s'(x)) = s(x)$.
- The strategy $\Phi(\tau)$ of Eloise in MEG(ϕ) is to make sure that at all times the position S consists only of C-translations of τ-positions.

$$
\begin{tikzcd}
C \arrow[r, dash] \arrow[d, dash, "\Phi(\tau)"'] & N \arrow[r, phantom, "\subseteq"] & M \arrow[d, dash, "\tau"] \\
\phi & & \phi
\end{tikzcd}
$$

Figure: From model to model existence.

Consistency $\Rightarrow$ model and truth

Theorem

Every strategy τ of Eloise in MEG(ϕ) *determines a model $M(\tau)$ and a strategy $\Psi(\tau)$ of Eloise in* $\boldsymbol{G(M(\tau),\phi)}$. *If τ is winning, then so is $\Psi(\tau)$.*

[Beth, 1955]

$$\begin{tikzcd}[column sep=large, row sep=large]
C \arrow[r, dash] \arrow[d, dash, "\tau"'] & M(\tau) \arrow[d, dash, "\Psi(\tau)"] \\
\phi & \phi
\end{tikzcd}$$

Figure: From model existence to a model.

Let σ_0 be the following enumeration strategy of Abelard in MEG(ϕ): During the game Abelard makes sure that if S is the position, then:

1. If $(\psi_0 \wedge \psi_1, s) \in S$, then during the game he will at some position $S' \supseteq S$ decide that the next position is $S' \cup \{(\psi_0, s)\}$ and at some further position $S'' \supseteq S'$ he will decide that the next position is $S'' \cup \{(\psi_1, s)\}$.
2. If $(\psi_0 \vee \psi_1, s) \in S$, then at some position $S' \supseteq S$ Abelard asks Eloise to choose whether the next position is $S' \cup \{(\psi_0, s)\}$ or $S' \cup \{(\psi_1, s)\}$.
3. If $(\forall x\theta, s) \in S$, then for all n during the game he will at some position $S' \supseteq S$ decide that the next position is $S' \cup \{(\theta, s(c_n/x)\}$.
4. If $(\exists x\theta, s) \in S$, then at some position $S' \supseteq S$ Abelard will ask Eloise to choose n after which the next position is $S' \cup \{(\theta, s(c_n/x))\}$.

- Let us play MEG(ϕ) while Abelard uses this strategy and Eloise plays τ.
- Let $\mathcal{S} = \langle S_n : n < \omega \rangle$ be the (unique) infinite sequence of positions during this play. Note that $S_n \subseteq S_{n+1}$ for all n. Let Γ be the union of all the positions in $\mathcal{S}$.
- We build a model $M = M(\tau)$ as follows[1] : The domain of the model is $\{c_n : n \in \mathbb{N}\}$. If R is a relation symbol, then we let $R(c_{n_0}, \ldots, c_{n_k})$ hold in M if $(R(x_{n_0}, \ldots, x_{n_k}), s) \in \Gamma$ for some s such that $s(x_i) = c_i$ for $i = n_0, \ldots, n_k$.
- The strategy $\boldsymbol{\Psi(\tau)}$ of Eloise in $G(M, \phi)$ is the following: She makes sure that if the position in $G(M, \phi)$ is (ψ, s), then $(\psi, s) \in \Gamma$.

[1] We assume ϕ has a relational vocabulary and does not contain the identity symbol.

- A winning strategy of Eloise in MEG(ϕ) can be conveniently given in the form of a so-called *consistency property* [Smullyan, 1963], which is just a set of finite sets of sentences satisfying conditions which essentially code a winning strategy for Eloise in MEG(ϕ).
- Sometimes it is more convenient to use a consistency property than Model Existence Game. But as far as strategies of Eloise are concerned, the two are one and the same thing.
- Consistency properties have been successfully used to prove interpolation and preservations results in model theory, especially infinitary model theory [Makkai, 1969].

- Suppose now Abelard has a winning strategy in MEG(ϕ).
- We can form a tree, a Beth Tableaux, of all the positions when Abelard plays his winning strategy and we stop playing as soon as Abelard has won.
- Every branch of the tree is finite and ends in a position which includes a contradiction.
- We can make the tree finite. We can then view this tree as a proof of $\neg\phi$. In this sense the Model Existence Game builds a bridge between proof theory and model theory.
- Strategies of Abelard direct us to proof theory, while strategies of Eloise direct us to model theory.

Apart from first order and infinitary logic, the Model Existence Game can be used in the proof theory and model theory of

- Propositional and modal logic.
- Logic with generalized quantifiers (using weak models, which have to be transformed to real models by a model theoretic argument [Keisler, 1970]).
- Higher order logic (using general models [Henkin, 1950]).
- Infinitary logic $L_{\kappa\lambda}$, (using chain models [Karp, 1974]).

EF (Ehrenfeucht-Fraïssé) game

- In the EF game we have a model but no sentence.
- The sentence should be true in one but false in the other. It may be that no such sentence can be found, i.e. the models are elementarily equivalent.
- In the EF game strategies of one player track possibilities for elementary equivalence and the strategies of the other player track possibilities for a separating sentence.
- [Fraïssé, 1954], [Ehrenfeucht, 1961]
- M and N are two structures for the same vocabulary L.

Definition

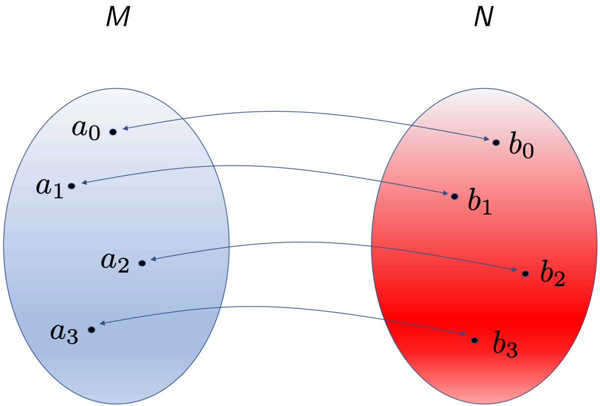
The game $\mathrm{EF}_m(M, N)$ has two players Abelard and Eloise and m moves. A position is a set

$$s = \{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\} \tag{1}$$

of pairs of elements such that the a_i are from M and the b_i are from N, and $n \leq m$. In the beginning the position is $\emptyset$. The rules:

1. Abelard may choose some $a_n \in M$. Then Eloise chooses $b_n \in N$ and the next position is $s \cup \{(a_n, b_n)\}$.
2. Abelard may choose some $b_n \in N$. Then Eloise chooses $a_n \in M$ and the next position is $s \cup \{(a_n, b_n)\}$.

Abelard wins if during the game the position (1) is such that $(a_0, \ldots, a_{n-1})$ satisfies some literal in M but $(b_0, \ldots, b_{n-1})$ does not satisfy the corresponding literal in N.


M
N
a_0
a_1
a_2
a_3
b_0
b_1
b_2
b_3

- Intuitively, Eloise defends the proposition that M and N are very similar.
- Abelard doubts this similarity.
- If Eloise knows an isomorphism $f : M \to N$ she can respond by playing always so that $b_n = f(a_n)$.
- Two models of (any) size $\geq m$ in the empty vocabulary.
- Two finite linear orders of (any) size $\geq 2^m$.
- This game is **determined**.
- How long games can Eloise win although $M \not\cong N$? Interesting for transfinite games.

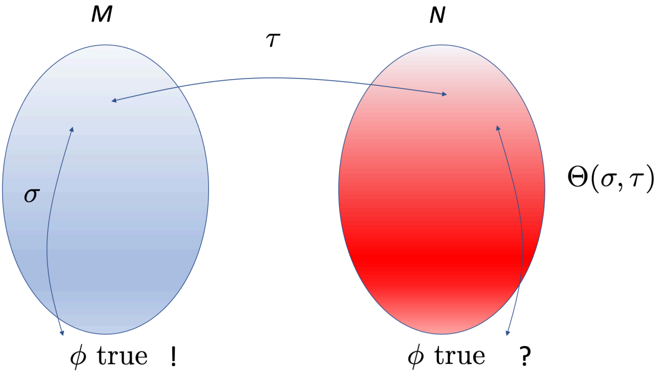
- A logician's version of isomorphism.
- A formula "is" this game.

Strategy of Eloise ⇒ elementary equivalence

Theorem

Suppose ϕ is an $L_{\infty\omega}$-sentence of quantifier rank $\leq m$. Every strategy τ of Eloise in $EF_m(M, N)$, and every strategy σ of Eloise in $G(M, \phi)$ determine a strategy $\Theta(\sigma, \tau)$ of Eloise in $G(N, \phi)$. If τ and σ are winning strategies, then so is $\Theta(\sigma, \tau)$.

[Ehrenfeucht, 1961]


M
N
τ
σ
$\Theta(\sigma, \tau)$
ϕ true !
ϕ true ?

- We call a position of the game $\mathrm{EF}_m(M, N)$ a **τ-position** if it arises while Eloise is playing τ.
- We call a position of the game $G(M, \phi)$ a **σ-position**, if it arises while Eloise is playing σ.
- If the position of the game $G(N, \phi)$ is (ψ, s), the **strategy** $\Phi(\sigma, \tau)$ of Eloise is to play simultaneously $G(N, \phi)$, $\mathrm{EF}_m(M, N)$ and $G(M, \phi)$, and make sure that if

$$\pi = \{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$$

is the current τ-position in $\mathrm{EF}_m(M, N)$ and $s(x) = \pi(s'(x))$ for all x in the domain of s, then (ψ, s') is the current σ-position in $G(M, \phi)$ (see Figure).

$$\begin{tikzcd}[arrows=dash]
M \arrow[r, "\pi"] \arrow[d, "s'"'] & N \arrow[d, "s"] \\
\phi & \phi
\end{tikzcd}$$

Figure: The strategy $\Theta(\sigma, \tau)$,

- There is a tight connection between σ, τ and $\Theta(\sigma, \tau)$. This is reflected in a connection between ϕ and $\mathrm{EF}_m(M, N)$.
- If the non-logical symbols of ϕ are in $L' \subset L$, then it suffices that τ is a strategy of Eloise in the game $\mathrm{EF}_m(M \restriction L', N \restriction L')$ between the reducts $M \restriction L'$ and $N \restriction L'$.
- If we know more about the syntax of ϕ, for example that it is existential, universal or positive, we can modify $\mathrm{EF}_M(M, N)$ accordingly by stipulating that Abelard only moves in M, only moves in N, or that he has to win by finding an atomic (rather than literal) relation which holds in M but not in N.
- Winning strategies for the EF game are a standard method for showing that certain kinds of sentences do not exist.

Strategy of Abelard $\Rightarrow$ separating sentence

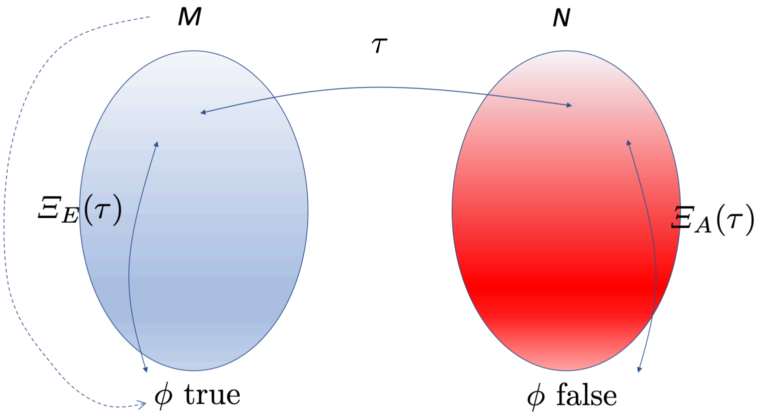
Theorem

Suppose M and N are models of the same vocabulary and $m < \omega$.

1. *There is a sentence $\phi \in L_{\infty\omega}$ of quantifier rank $\leq m$ and mappings Ξ_E and Ξ_A such that if τ is a strategy of Abelard in $EF_m(M, N)$, then $\Xi_E(\tau)$ is a strategy of Eloise in $G(M, \phi)$, and $\Xi_A(\tau)$ is a strategy of Abelard in $G(N, \phi)$.*
2. *If τ is a winning strategy, then $\Xi_E(\tau)$ and $\Xi_A(\tau)$ are winning strategies.*

Note: If L is finite and relational, the sentence ϕ is logically equivalent to a first order sentence of quantifier rank $\leq m$.

[Ehrenfeucht, 1961]


M
N
τ
$\Xi_E(\tau)$
$\Xi_A(\tau)$
ϕ true
ϕ false

Suppose s is an assignment into M with domain $\{x_0, \ldots, x_{n-1}\}$.
Let

$$\begin{aligned}
\psi^{0,n}_{M,s} &= \bigwedge_i \psi_i \\
\psi^{m+1,n}_{M,s} &= (\forall x_n \bigvee_{a \in M} \psi^{m,n+1}_{M,s(a/x_n)}) \wedge (\bigwedge_{a \in M} \exists x_n \psi^{m,n+1}_{M,s(a/x_n)}),
\end{aligned}$$

where ψ_i lists all the literals in the variables $x_0, \ldots, x_{n-1}$ satisfied by s in M.
The sentence ϕ we need is $\boldsymbol{\psi^{m,0}_{M,\emptyset}}$.

- Clearly Eloise has a trivial strategy $\Xi_E(\tau)$ in $G(M, \phi)$ (independently of τ), and this strategy is always a winning strategy.
- We now describe the strategy $\Xi_A(\tau)$ of Abelard in $G(N, \phi)$.
- We call a position of the EF-game a **τ-position** if it arises while Abelard is playing τ.
- Suppose s is an assignment into M and s' an assignment into N, both with domain $\{x_0, \ldots, x_{n-1}\}$. We use $\boldsymbol{s \cdot s'}$ to denote the set of pairs $(s(x_i), s'(x_i))$, $i = 0, \ldots, n-1$. The **strategy** of Abelard is to play $G(N, \phi)$ in such a way that if the position at any point is $(\psi^{i,m-i}_{M,s}, s')$, then $s \cdot s'$ is a τ-position.

- If τ is a winning strategy of Abelard even in the game $\mathrm{EF}_m(M \restriction L', N \restriction L')$ for some $L' \subset L$, then the separating sentence ϕ can be chosen so that its non-logical symbols are all in L'.
- If τ is such that Abelard plays only in M, we can make ϕ existential.
- If τ is such that Abelard plays only in N, we can make ϕ universal.
- If Abelard wins with τ even the harder game in which he has to win by finding an atomic (rather than literal) relation which holds in M but not in N, then we can take ϕ to be a positive sentence.

- Strategies in $\mathrm{EF}_m(M, N)$ also reflect structural properties of M and N.
- If we know a strategy of Eloise in $\mathrm{EF}_m(M_i, N_i)$ for $i \in I$, we can construct strategies of Eloise for EF games between products and sums of the models M_i and the respective products and sums of the models N_i. This can be extended to so-called κ-local functors [Feferman, 1972]. For an example of the use of tree-decompositions, see e.g. [Grohe, 2007].
- EF games are known for infinitary logics, generalized quantifiers, and higher order logics.
- In modal logic the corresponding game is called the bisimulation game.

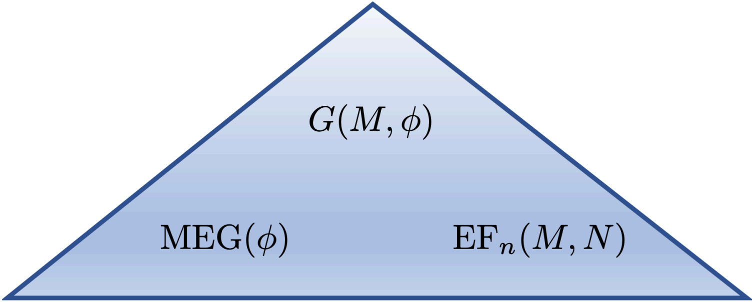
- EF game for propositional logic [Hella and Väänänen, 2015].
- EF-game for $L_{\omega_1\omega}$ [Väänänen and Wang, 2013].
- An EF-game with "delay" and a Lindström Theorem[2] for a new infinitary logic between $L_{\kappa\omega}$ and $L_{\kappa\kappa}$ [Shelah, 2012].

[2] A Lindström Theorem is a semantic characterization of a logic.

Summary

- The three games incorporate everything important in logic: truth, consistency, formula, proof, structure, the human user, etc etc
- A coherent uniform approach to syntax and semantics, to model theory and proof theory.
- The Evaluation Game and the EF game are oblivious to whether the models are finite or infinite.
- There is a lot of potential for the study of the interaction between the three games, the Strategic Balance of Logic.

Thank you!



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