The derivator of setoids

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Outline

Constructive homotopy theory

Question

What does homotopy theory look like in constructive mathematics? Or even in ZF set theory without the axiom of choice?

This question has two sides:

1. The formal aspects of homotopy theory
2. The behavior of specific homotopy theories, and particularly "the" homotopy theory of "spaces".

Let's discuss the second one first.

The homotopy theory of spaces

Classically, there are many different presentations of the homotopy theory of spaces:

- Topological spaces, up to weak homotopy equivalence
- Simplicial sets (Kan complexes)
- Semi-simplicial sets
- Presheaves on other "test categories"
- Small categories, up to nerve equivalences
- ...

(We say "spaces", but it's often better to think of this as a homotopy theory of ∞-groupoids.)

Constructive homotopy spaces

At least two of these now have constructive versions:

- Simplicial sets (Henry, Gambino, Sattler, Szumilo)
- Equivariant cartesian cubical sets (Awodey, Cavallo, Coquand, Riehl, Sattler)
- Effective/uniform simplicial sets? (van den Berg, Faber)

However, it's not known if these homotopy theories are constructively equivalent to each other.

If they are not, then which is "correct"?

What does that even mean?

A proposed answer

Recall that the 1-category **Set** is the "free cocompletion of a point": every object $M \in \mathscr{C}$ of a cocomplete category $\mathscr{C}$ determines a unique cocontinuous functor $\mathbf{Set} \to \mathscr{C}$ sending $*$ to M.

Proposal

The homotopy theory of spaces should be the free cocompletion of a point among homotopy theories.

This is true classically, so it's reasonable to want constructively. And it's a universal property, so it characterizes an object up to equivalence (if such an object exists).

But what does it mean precisely?

What is a homotopy theory?

This leads us back to the first question.

What is a homotopy theory, anyway?

Classical answer

A Quillen model category.

- Basic parts of model category theory work constructively, when formulated with specified operations.
- More advanced parts, like the small object argument, seem to depend on choice.
- Hard to formulate universal properties of homotopy theories with model categories.
- Classically, most interesting homotopy theories admit model category presentations, but we shouldn't assume this will necessarily be true constructively.

What is a homotopy theory? Take two.

Modern answer

An $(\infty, 1)$-category.

(i.e. a higher category with n-morphisms for all n, invertible if $n > 1$.)

- In classical mathematics, now the "standard" way to formulate such properties.
- Requires picking a model: quasicategories, simplicial categories, complete Segal spaces, etc. . .
- None of these models has yet been developed constructively.
- Finding the "correct" constructive notion of $(\infty, 1)$-category seems likely to be at least as hard as the problem for ∞-groupoids (i.e. spaces)!

What is a homotopy theory? Take three.

Quasi-modern answer (Heller, Grothendieck, Franke)

A derivator.

- A "quotient" rather than a presentation, so it should always exist, even constructively.
- Uses only 1-categories, which we understand constructively.

> *"Derivators ... give us the language to characterize higher category theory using only usual category theory, without any emphasis on any particular model (in fact, without assuming we even know any)."* – Denis-Charles Cisinski

To explain what a derivator is, let's consider an example...

Outline

Exact completion

Definition

A category is (Barr-)exact if it has finite limits and every internal equivalence relation has a quotient of which it is the kernel.

The forgetful functor

$$\mathsf{ExactCat} \to \mathsf{FiniteLimitCat}$$

has a left adjoint, the free exact completion

$$\mathscr{E} \mapsto \mathscr{E}_{\mathsf{ex}}$$

Pseudo equivalence relations

Let $\mathscr{E}$ have finite limits.

Definition

A pseudo equivalence relation in $\mathscr{E}$ is a span $X_0 \xleftarrow{s} X_1 \xrightarrow{t} X_0$ together with:

- A reflexivity map $r : X_0 \to X_1$ with $sr = tr = 1_{X_0}$.
- A symmetry map $v : X_1 \to X_1$ with $sv = t$ and $tv = s$.
- A transitivity map $t : X_1 \times_{X_0} X_1 \to X_1$ with ...

- An equivalence relation iff $(s, t) : X_1 \to X_0 \times X_0$ is monic.
- Pseudo equivalence relations in **Set** are also called setoids.

The exact completion

Definition

For pseudo-equivalence relations X, Y a morphism representative $f : X \to Y$ is

- $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ with $sf_1 = f_0 s$ and $tf_1 = f_0 t$.

Two morphism representatives f, g are equivalent if

- There exists $h : X_0 \to Y_1$ with $sh = f_0$ and $th = g_0$.

The free exact completion $\mathscr{E}_{\mathrm{ex}}$ consists of:

- Pseudo equivalence relations, with
- Equivalence classes of morphism representatives.

Remark

$\mathbf{Set} \hookrightarrow \mathbf{Set}_{\mathrm{ex}}$ is an equivalence $\iff$ the axiom of choice holds.

Limits in the exact completion

Since $\mathscr{E}_{\mathrm{ex}}$ is supposed to be exact, it must have finite limits.

Example

The product of $X, Y \in \mathscr{E}_{\mathrm{ex}}$ has

$$(X \times Y)_0 = X_0 \times Y_0 \qquad (X \times Y)_1 = X_1 \times Y_1.$$

Given morphisms $Z \to X$ and $Z \to Y$ in $\mathscr{E}_{\mathrm{ex}}$, choose representatives f and g. Then $(f_0, g_0) : Z_0 \to X_0 \times Y_0$ and $(f_1, g_1) : Z_1 \to X_1 \times Y_1$ represent a morphism $Z \to X \times Y$.

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In the infinite case, we can say:

Theorem

If $\mathscr{E}$ is complete and the axiom of choice holds, $\mathscr{E}_{\text{ex}}$ is also complete.

Infinite "limits"

And yet, $\mathscr{E}_{\mathrm{ex}}$ obviously does have a sort of "product" operation:

$$\left(\prod_a X_a\right)_0 = \prod_a X_{a,0} \qquad \left(\prod_a X_a\right)_1 = \prod_a X_{a,1}$$

This has a "universal property" relative to cones of morphism representatives, rather than the usual cones of morphisms.

How can we formulate this abstractly?

1. Regard $\mathscr{E}_{\mathrm{ex}}$ as the quotient of an "E-category".
2. Extend $\mathscr{E}_{\mathrm{ex}}$ to a derivator!

Coherent diagrams

Let A be a small category.

Definition

A coherent A-diagram in $\mathscr{E}_{\mathrm{ex}}$ comprises

- A pseudo equivalence relation $X_a \in \mathscr{E}_{\mathrm{ex}}$ for each object $a \in A$.
- A morphism representative $X_\alpha : X_a \to X_{a'}$ for each morphism $\alpha : a \to a'$ in A.
- Specified witnesses $X_{a,0} \to X_{a'',1}$ of functoriality.

A morphism representative of such, $f : X \to Y$, comprises

- A morphism representative $f_a : X_a \to Y_a$ for each $a \in A$.
- Specified witnesses $X_{a,0} \to Y_{a',1}$ of naturality.

Two such are equivalent if there are witnesses of such, $X_{a,0} \to Y_{a,1}$. This defines a category $\mathscr{E}_{\mathrm{ex}}(A)$.

Infinite "limits", derivator-style

Theorem

If $\mathscr{E}$ is complete, then each "constant coherent diagram" functor $\mathrm{const} : \mathscr{E}_{\mathsf{ex}} \to \mathscr{E}_{\mathsf{ex}}(A)$ *has a right adjoint.*

Theorem

If $\mathscr{E}$ has small pullback-stable coproducts, then each $\mathrm{const} : \mathscr{E}_{\mathsf{ex}} \to \mathscr{E}_{\mathsf{ex}}(A)$ *has a left adjoint.*

Outline

Derivators

Let **Cat** and **CAT** be the 2-categories of small and large categories.

Definition (Constructive version)

A derivator is a 2-functor $\mathscr{D} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ such that:

1. $\mathscr{D}$ takes finite coproducts to products.
2. For $A \in \mathbf{Cat}$, the functor $\iota^* : \mathscr{D}(A) \to \mathscr{D}(A_0)$, with A_0 the discrete category on the objects of A, is conservative.
3. For $u : A \to B$ in $\mathbf{Cat}$, the functor u^* has a left adjoint $u_!$ and a right adjoint u_*.
4. Comma squares in **Cat** satisfy the Beck-Chevalley condition for these adjoints.

Examples of derivators

Example

If $\mathscr{E}$ is complete and has small pullback-stable coproducts, then $\mathscr{E}_{\mathrm{ex}}$ is a derivator, with $\mathscr{E}_{\mathrm{ex}}(A)$ the category of coherent diagrams.

Example

If $\mathscr{C}$ is a complete and cocomplete category, it defines a derivator by $\mathscr{C}(A) = \mathscr{C}^A$, the usual functor category. The functors $u_!$ and u_* are pointwise Kan extensions.

Example (in classical mathematics)

Every complete and cocomplete homotopy theory $\mathscr{M}$ (= model category or $(\infty, 1)$-category) determines a derivator, where $\mathscr{M}(A)$ is the homotopy category of homotopy-coherent A-diagrams in $\mathscr{M}$.

The 2-category of derivators

Definition

A morphism of derivators $G : \mathscr{D}_1 \to \mathscr{D}_2$ is a pseudonatural transformation. It is cocontinuous if for any $u : A \to B$, the mate

$$u_! G_A \to G_B u_!$$

is an isomorphism.

A transformation of derivators is a modification. This defines

$$\mathrm{Hom}(\mathscr{D}_1, \mathscr{D}_2) \qquad \text{and} \qquad \mathrm{Hom}_{\mathrm{cc}}(\mathscr{D}_1, \mathscr{D}_2)$$

The free cocompletion of a point

Theorem (Heller, Cisinski – in classical mathematics)

The derivator of spaces $\mathscr{S}$ is the free cocompletion of a point. That is, for any derivator $\mathscr{D}$, evaluation at $* \in \mathscr{S}(\mathbf{1})$:

$$\mathrm{Hom}_{\mathrm{cc}}(\mathscr{S}, \mathscr{D}) \to \mathscr{D}(\mathbf{1})$$

is an equivalence of categories.

- This is a 2-categorical universal property, expressed by an equivalence of 1-categories. No more higher-categorical machinery than the analogous fact for the 1-category **Set**.
- The bare homotopy category $\mathscr{S}(\mathbf{1})$ is not structured enough to even state, let alone prove, such a property.

The constructive homotopy theory of spaces?

Proposal

The "correct" homotopy theory of spaces, constructively, should define a derivator $\mathscr{S}$ that is the free cocompletion of a point.

- It remains to be seen whether such a thing exists!
- We can try to understand what it would be like, by studying derivators that would be localizations or subcategories of it.

Outline

Some classical subcategories of spaces

Classically, the homotopy theory $\mathscr{S}$ of spaces has many interesting subcategories, notably:

- The category **Set** of sets, a.k.a. homotopy 0-types
- The 2-category **Gpd** of groupoids, a.k.a. homotopy 1-types
- The $(n+1)$-category n-**Typ** of homotopy n-types
- The poset **Prop** of truth values (the case $n = -1$)

Each of these is the free cocompletion of a point in “its own world”:

- **Set** among 1-categories (i.e. $(1,1)$-categories)
- **Gpd** among $(2,1)$-categories
- n-**Typ** among $(n+1, 1)$-categories
- **Prop** among posets (i.e. “$(0,1)$-categories”)

Identifying truncated derivators

Q: How can we tell whether a derivator $\mathscr{D}$ "is" an $(n,1)$-category rather than a general $(\infty,1)$-category?

A: By the behavior of (co)limits of constant diagrams.

Example

In a 1-category, the equalizer of a constant diagram

$$M \underset{1_M}{\overset{1_M}{\rightrightarrows}} M$$

is just the object M again. But in a higher category, such a (homotopy) equalizer is the free loop space object LM.

Example

In a poset, the binary product of an object M with itself is just M again; but not in a general 1-category.

$\mathscr{D}$-equivalences

Let $\mathscr{D}$ be a derivator, and $p_A : A \to \mathbf{1}$ the projection for $A \in \mathbf{Cat}$.

Definition

$u : A \to B$ in **Cat** is a $\mathscr{D}$-equivalence if for any $M \in \mathscr{D}(\mathbf{1})$, it induces an isomorphism between limits of constant diagrams:

$$(p_B)_* (p_B)^* M \xrightarrow{\sim} (p_A)_* (p_A)^* M.$$

Example

If $\mathscr{D}$ is a 1-category, $(p_A)_* (p_A)^* M$ is the power $M^{\pi_0(A)}$ by the set of connected components of A. Thus, u is a $\mathscr{D}$-equivalence if it induces $\pi_0(A) \xrightarrow{\sim} \pi_0(B)$. The converse holds if $\mathscr{D} = \mathbf{Set}$.

More $\mathscr{D}$-equivalences

Example

u is a **Gpd**-equivalence iff it induces an equivalence of groupoid reflections, $\Pi_1(A) \xrightarrow{\sim} \Pi_1(B)$.

Example

u is a **Prop**-equivalence iff whenever B has an object, so does A.

Example (in classical mathematics)

u is an $\mathscr{S}$-equivalence iff it induces a weak homotopy equivalence of nerves.

And similarly for the derivators n-**Typ** of homotopy n-types, including **Gpd** $=$ 1-**Typ** and **Set** $=$ 0-**Typ** and **Prop** $= (-1)$-**Typ**.

Local derivators

Let $\mathscr{D}$ and $\mathscr{T}$ be derivators.

Definition

$\mathscr{D}$ is $\mathscr{T}$-local if every $\mathscr{T}$-equivalence is a $\mathscr{D}$-equivalence.

Examples

- Every complete and cocomplete 1-category is **Set**-local.
- Every complete lattice is **Prop**-local.
- In classical mathematics, every derivator is $\mathscr{S}$-local and every $(n+1,1)$-category is n-**Typ**-local.

Relative free cocompletions

Definition

$\mathscr{T}$ is a relative free cocompletion of a point if for any $\mathscr{T}$-local derivator $\mathscr{D}$, evaluation at $* \in \mathscr{T}(\mathbf{1})$:

$$\mathrm{Hom}_{\mathrm{cc}}(\mathscr{T}, \mathscr{D}) \to \mathscr{D}(\mathbf{1})$$

is an equivalence of categories.

Examples

Set, **Prop**, and **Gpd** are relative free cocompletions of a point.

Examples (in classical mathematics)

n-**Typ** is a relative free cocompletion of a point, for all $n \geq -1$.

BUT

Set$_{\mathrm{ex}}$ **is NOT Set-local!**

Even though it is intuitively still "1-categorical".

Observation

$u : A \to B$ is a **Set**$_{\mathrm{ex}}$-equivalence iff it induces an isomorphism of setoids of connected components, $\pi_0^e A \xrightarrow{\simeq} \pi_0^e B$:

- $(\pi_0^e A)_0 =$ the set of objects of A.
- $(\pi_0^e A)_1 =$ the set of zigzags of morphisms in A.

This is a stronger condition than being a **Set**-equivalence.

Example

Let $p : X \to Y$ be a surjection, make Y a discrete category, and X a category such that p becomes fully faithful. Then p is a **Set**-equivalence, but not a **Set**$_{\mathrm{ex}}$-equivalence unless p has a section.

More relative free cocompletions

Theorem

$\mathbf{Set}_{\mathrm{ex}}$ *is also a relative free cocompletion of a point.*

- $\mathbf{Set}$ is $\mathbf{Set}_{\mathrm{ex}}$-local, though $\mathbf{Set}_{\mathrm{ex}}$ is not $\mathbf{Set}$-local.
- $\mathscr{E}_{\mathrm{ex}}$ is $\mathbf{Set}_{\mathrm{ex}}$-local for any suitable $\mathscr{E}$.

What does this tell us?

Conclusion

If there is a derivator $\mathscr{S}$ that is an "absolute" free cocompletion of a point constructively, its subcategory of "0-types" must contain not just sets, but also setoids.

- Existing simplicial ~~and cubical~~ proposals do have this property.
- But it's disturbing for the prospect of applying homotopy theory to deduce conclusions about sets (or sheaves of sets)!
- Related problem: defining "realizability higher toposes" whose 0-types form a realizability 1-topos.

What can we do?

Possible responses:

1. Bite the bullet: learn to pass back and forth all the time between setoids and sets.
2. No sets, only setoids: use a foundation of "pre-sets", like MLTT without identity types.
3. Spaces are primitive: use a foundation like HoTT, where spaces are basic objects, not defined out of sets.
4. Don't give up: is there a principled way to exclude $\mathbf{Set}_{\mathrm{ex}}$?

A fragment of the locality preorder

Towards anafunctors

There are already grounds on which to object to **Cat** and **Gpd**: they do not invert weak equivalences of categories (fully faithful and essentially surjective functors).

Definition (Makkai)

An anafunctor is a morphism in the category of fractions of **Cat** with respect to weak equivalences.

- Is there a derivator $\mathbf{Gpd}_{\mathbf{ana}}$ of "groupoids and anafunctors" that sits directly over **Set**? Or even $\mathscr{S}_{\mathbf{ana}}$?

  (May require a weak form of choice like SCSA/WISC/AMC.)
- Is there a notion of "ana-derivator" using $\mathbf{Cat}_{\mathbf{ana}}$ instead of **Cat**, that includes only the right-hand column?

More conjecture: towards spaces