Towards a point-free model theory

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Traditional model theory uses signatures and formulas to specify structures and their morphisms. Structures are specified by theories and, as morphisms, one takes homomorphisms, embeddings, elementary embeddings, etc.

[Shelah 1987] introduced abstract elementary classes using signatures but no formulas. Instead, structures and morphisms are governed by abstract properties.

[Makkai, Paré 1989] defined accessible categories as a framework for model theory without signatures and without underlying sets. They showed that these categories correspond to infinitary theories, i.e., theories in $L_{\kappa\lambda}$ using disjunctions of $< \kappa$ formulas and quantifications over $< \lambda$ variables.

This means that every accessible category is equivalent to the category of models of an $L_{\kappa\lambda}$-theory with homomorphisms as morphisms and, conversely, every category of models of an $L_{\kappa\lambda}$-theory with $L_{\kappa\lambda}$-elementary embeddings as morphisms is accessible.

Definition 1. A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-directed colimits,

(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$.

An object A is λ-presentable if its hom-functor

$$\hom(A, -) : \mathcal{K} \to \textbf{\textit{Set}}$$

preserves λ-directed colimits.

A category is *accessible* if it is λ-accessible for some regular cardinal λ.

Every AEC is an accessible category with directed colimits and morphisms being monomorphisms (Beke, Lieberman, LR 2012).

Every accessible category can be equipped with underlying sets and formulas but it can be done in many ways. My aim is to present some advantages of this "point-free" approach.

Every object of an accessible category is λ-presentable for some regular cardinal λ. The smallest regular cardinal λ such that A is λ-presentable is called the *presentability rank* of A. If the presentability rank of A is μ^+ then μ is called the *size* of A.

The size of an infinite set is its cardinality. The same for an infinite poset or for an uncountable group.

In an AEC, sizes eventually coincide with cardinalities of underlying sets.

The size of an infinite complete metric space is its density character, i.e., the smallest cardinality of a dense subset. The same for infinite dimensional Banach spaces. The size of an infinite dimensional Hilbert space is the cardinality of its orthonormal base. Let us add that ***Ban*** and ***Hilb*** (with linear maps of norm ≤ 1) are $\aleph_1$-accessible with directed colimits.

Proposition 1. (Beke, JR 2012) In every accessible category with directed colimits, presentability ranks are successors starting from some cardinal.

In a general accessible category, it is true under GCH.

Any infinite-dimensional Banach space has cardinality $\lambda^{\aleph_0}$ for some infinite cardinal λ [Bartoszyński, Džamonja, Halbeisen, Murtinová, Plichko 2005]. Thus there are no Hilbert spaces in cardinality λ of countable cofinality. But there are Hilbert spaces of any infinite size. Thus in ***Hilb*** sizes never start to coincide with cardinalities and there are arbitrarily large gaps of cardinalities but not in sizes.

In an abstract elementary class, there are not arbitrarily large gaps of sizes.

Problem 1. (Beke, JR 2012) Are there accessible categories (with directed colimits) with arbitrarily large gaps of sizes?

An accessible category is called *LS-accessible* if this cannot happen.

Proposition 2. (Beke, JR 2012) Every accessible category $\mathcal{K}$ with directed colimits equipped with a faithful functor to ***Set*** preserving directed colimits is LS-accessible.

Proposition 3. (Lieberman, JR 2014) Every accessible category $\mathcal{K}$ with directed colimits whose morphisms are monomorphisms is LS-accessible.

Every AEC is equipped with a faithful functor to ***Set*** preserving directed colimits.

Let $\boldsymbol{Ban}_0$, or $\boldsymbol{Hilb}_0$ be the categories of Banach, or Hilbert spaces and linear isometries. They are $\aleph_1$-accessible categories with directed colimits whose morphisms are monomorphisms.

Proposition 4. (Lieberman, JR, Vasey 2019) There is no faithful functor $\boldsymbol{Hilb}_0 \to \boldsymbol{Set}$ preserving directed colimits.

Hence $\boldsymbol{Hilb}_0$ is not equivalent to any AEC. The same for $\boldsymbol{Ban}_0$ or $\boldsymbol{CAlg}_0$ (commutative C^*-algebras and embeddings).

$L_{\kappa\omega}$-theories yield (∞, ω)-elementary categories which are AECs. It is easy to give an AEC which is not $L_{\kappa\omega}$-axiomatizable in its signature. For instance, the category $\boldsymbol{Set}_{un}$ of uncountable sets and monomorphisms.

Theorem 1. (Henry 2018) $\boldsymbol{Set}_{un}$ is an abstract elementary class which is not (∞, ω)-elementary.

The proof is much more difficult because one has to exclude all possible faithful functors to ***Set*** preserving directed colimits. We only know that the natural one does not work. With Makkai, we failed to prove it. I will return to this later.

Let $\mathcal{K}$ be an accessible category with directed colimits and λ an infinite cardinal. $\mathcal{K}$ is λ-*categorical* if it has, up to isomorphism, precisely one object of size λ.

Shelah's Categoricity Conjecture claims that for every AEC $\mathcal{K}$ there is a cardinal κ such that $\mathcal{K}$ is either λ-categorical for all $\lambda \geq \kappa$ or $\mathcal{K}$ is not λ-categorical for all $\lambda \geq \kappa$.

This was conjectured by Loś for first-order theories in a countable language in 1954 and proved by Morley in 1965. In 1970, Shelah extended it for uncountable languages. SCC is the main test question for AECs.

Of course, SCC was formulated using external sizes, i.e., cardinalities of underlying sets. Since they coincide with internal sizes starting from some cardinal, SCC is the property of the category $\mathcal{K}$.

Problem 2. Can Shelah's Categoricity Conjecture be extended to accessible categories with directed colimits?

Example 1.(Beke, JR 2012) Let $\mathcal{K}$ be an accessible category with directed colimits which is not LS-accessible. Then $\mathcal{K} \amalg \boldsymbol{Set}$ is an accessible category with directed colimits which does not satisfy SCC.

To deal with SCC, Shelah introduced many tools including stable independence $\downarrow$ generalizing linear independence in vector spaces and algebraic independence in fields. In [Lieberman, JR, Vasey 2019], we showed that this is a property of an accessible category, i.e., a point-free concept.

The *stable independence* $\downarrow$ in $\mathcal{K}$ consists in the choice of squares

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

which are declared to be independent. We say that M_1 and M_2 are independent over M_0 in M_3.

There are seven axioms, for instance, every span can be completed to an independent square and uniquely up to an equivalence, independent squares are closed under composition, etc. But the crucial axiom is that the category $\mathcal{K}_{\downarrow}$ whose objects are morphisms in $\mathcal{K}$ and morphisms are independent squares is accessible. This reflects model-theoretical witness and local character properties. Without this axiom we speak about *weak stable independence*.

Theorem 2. (Lieberman, JR, Vasey 2019) An accessible category with directed colimits has at most one stable independence.

Every AEC having stable independence is stable and tame. These two properties are defined using Galois types, i.e., pairs (f, a) where $f : M \to N$ is a morphism and $a \in UN$ an element of the underlying set of N. I do not know whether stability and tameness are point-free concepts.

Every accessible category with pushouts has stable independence consisting of pushout squares. But accessible categories whose morphisms are monomorphisms do not have pushouts.

A *cellular* category is a cocomplete category equipped with a class $\mathcal{M}$ of morphisms containing all isomorphisms and closed under pushouts and transfinite compositions.

A commutative square

$$\begin{array}{ccc} M_1 & \rightarrow & M_3 \\ \uparrow & & \uparrow \\ M_0 & \rightarrow & M_2 \end{array}$$

is called *cellular* if the induced morphism $P \to M_3$ from the pushout belongs to $\mathcal{M}$.

Proposition 5. In every cellular category, cellular squares form a weak stable independence.

Choosing $\mathcal{M} = \mathrm{Iso}$, cellular squares are pushout squares.

A cellular category is *combinatorial* if it is cofibrantly generated, i.e., if $\mathcal{M}$ is a closure of a set of morphisms under pushouts, transfinite compositions and retracts. This terminology is borrowed from homotopy theory because in a combinatorial model category both cofibrations and trivial cofibrations are cofibrantly generated.

In a cellular category $(\mathcal{K}, \mathcal{M})$, $\mathcal{K}_{\mathcal{M}}$ denotes the category of $\mathcal{K}$-objects and $\mathcal{M}$-morphisms.

A cellular category $(\mathcal{K}, \mathcal{M})$ is called

1. *coherent* if $gf, g \in \mathcal{M}$, then $f \in \mathcal{M}$,
2. λ-*continuous* if $\mathcal{K}_{\mathcal{M}}$ is closed under λ-directed colimits in $\mathcal{K}$,
3. λ-*accessible* if it is λ-continuous and both $\mathcal{K}$ and $\mathcal{K}_{\mathcal{M}}$ are λ-accessible,
4. *accessible* if it is λ-accessible for some λ.

Theorem 3. (Lieberman, JR, Vasey) Let $(\mathcal{K}, \mathcal{M})$ be an accessible cellular category which is retract-closed, coherent and $\aleph_0$-continuous. Then the following are equivalent:

1. $\mathcal{K}_{\mathcal{M}}$ has a stable independence,
2. cellular squares for a stable independence in $\mathcal{K}_{\mathcal{M}}$,
3. $(\mathcal{K}, \mathcal{M})$ is combinatorial.

Examples 2. Let $\mathcal{K}$ be locally presentable (= accessible and cocomplete.

(1) $\mathcal{K}, \mathcal{K}^2)$ is combinatorial and independent squares are commutative squares.

(2) $(\mathcal{K}, \mathit{RegMono})$ is cellular. If $\mathcal{K}$ is coregular, cellular squares are effective pullback squares of [Barr 1988]. If $\mathcal{K}$ has effective unions (= every pullback square is effective), $(\mathcal{K}, \mathit{RegMono})$ is combinatorial. This implies the well-known fact that Grothendieck toposes and Grothendieck abelian categories have enough injectives.

$(\boldsymbol{Ban}, \mathit{RegMono})$ is not combinatorial because it is not stable, or regular injectives do not form an accessible category.

(3) If $\mathcal{K}$ is locally finitely presentable, $(\mathcal{K}, \mathit{PureMono})$ is cellular. Cellular squares are pure effective squares of [Borceux, JR 2007]. If $\mathcal{K}$ is additive, $(\mathcal{K}, \mathit{PureMono})$ is combinatorial [Lieberman, Positselski, JR, Vasey 2020].

If $F : \mathcal{K} \to \mathcal{L}$ is a colimit preserving functor from a locally presentable category $\mathcal{K}$ to a combinatorial category $\mathcal{L}$ then $\mathcal{K}$ is combinatorial w.r.t. the *left induced cellular* structure given by preimages [Makkai, JR 2013] . This result uses Lurie's good colimits.

If $\mathcal{L}$ is coherent and $\aleph_0$-continuous, then this follows from Theorem 3.

A cellular category $(\mathcal{K}, \mathcal{M})$ yields the cellular category $(\mathcal{K}^2, \mathcal{M}!)$ where $\mathcal{M}!$ consists of cellular squares.

Theorem 3. [Lieberman, JR, Vasey 2020] If $(\mathcal{K}, \mathcal{M})$ is coherent, $\aleph_0$-continuous and combinatorial then $(\mathcal{K}^2, \mathcal{M}!)$ has the same properties.

It can be iterated and we get a higher-order stable independence on $\mathcal{K}_{\mathcal{M}}$.

Let $\mathcal{K}$ be an accessible category with directed colimits and all morphisms monomorphisms equipped with a faithful functor $U : \mathcal{K} \to \textbf{\textit{Set}}$ preserving directed colimits and monomorphisms. Operations definable on $\mathcal{K}$ are natural transformations $U^n \to U$ (this goes back to [Lawvere 1963]) and relations definable on $\mathcal{K}$ are subfunctors of U^n preserving directed colimits [JR 1981].

In this way, we can test whether $\mathcal{K}$ is equivalent to an AEC.

Di Liberti and Henry (2018) assigned to an accessible category $\mathcal{K}$ with directed colimits the category $S(\mathcal{K})$ of all functors $\mathcal{K} \to \textbf{\textit{Set}}$ preserving directed colimits.

This category is a Grothendieck topos, and they call it the Scott topos of $\mathcal{K}$ because it generalizes the usual Scott topology on a directed complete poset.

Conversely, to any Grothendieck topos $\mathcal{T}$ we can assign the category $P(\mathcal{T})$ of *points*, i.e., functors $\mathcal{T} \to \textbf{\textit{Set}}$ preserving colimits and finite limits. The category $P(\mathcal{T})$ is (∞, ω)-elementary. Moreover, (∞, ω)-elementary categories are precisely categories of points of Grothendieck toposes.

If $\boldsymbol{Acc}_\omega$ is the category of accessible categories with directed colimits and functors preserving directed colimits and $\boldsymbol{GTop}$ the category of Grothendieck toposes and functors preserving colimits and finite limits (geometric morphisms) then $S : \boldsymbol{Acc}_\omega \to \boldsymbol{GTop}$ is left adjoint to $P : \boldsymbol{GTop} \to \boldsymbol{Acc}_\omega$. This is the *Scott adjunction* of [Di Liberti, Henry 2018].

In particular, $S(\mathcal{K})$ bears knowledge about all finitary function and relation symbols definable in $\mathcal{K}$. Thus it plays the role of the "full (∞, ω)-theory" of $\mathcal{K}$.

The unit $\eta_\mathcal{K} : \mathcal{K} \to PS(\mathcal{K})$ maps $\mathcal{K}$ to the category of models of its full (∞, ω)-theory.

$\eta_\mathcal{K} : \mathcal{K} \to PS(\mathcal{K})$ is faithful iff there is a faithful functor $\mathcal{K} \to \boldsymbol{Set}$ preserving directed colimits.

If $\mathcal{K}$ is (∞, ω)-elementary, there is the "reduct" functor $R : PS(\mathcal{K}) \to \mathcal{K}$ such that $R\eta_\mathcal{K} = \mathrm{Id}_\mathcal{K}$.

Conversely, the existence of such a splitting of $\eta_\mathcal{K}$ makes $\mathcal{K}$ (∞, ω)-elementary. In this way, Henry proved Theorem 1. In fact $PS(\boldsymbol{Set}_{\aleph_1}) = PS(\boldsymbol{Set}_{\aleph_0}) = \boldsymbol{Set}_{\aleph_0}$.

Scott adjunction seems to be a strong tool for studying accessible categories with directed colimits.

[Di Liberti 2020] related Scott toposes to classifying toposes and [Espindola 2020] used this to deal with Problem 2.

Theorem 4. (Espindola 2020) Assume GCH and the existence of a proper class of strongly compact cardinals. Then every accessible category with directed colimits and interpolation satisfies SCC.

Interpolation is a strengthening of amalgamation and, under GCH, it is equivalent to SCC in AECs (Espindola 2018).

Propositions 2 and 3 follow from the existence of a faithful functor $E : \textbf{\textit{Lin}} \to \mathcal{K}$ preserving directed colimits where ***Lin*** is the category of linearly ordered set and strictly monotone maps. The reason is that ***Lin*** is LS-accessible and E preserves sizes.

This goes back to [Makkai, Paré] who observed that [Morley 1965] ensures that E exists for every (∞, ω)-elementary $\mathcal{K}$. It yields Ehrenfeucht-Mostowski models given by indiscernibles.

Let $\mathcal{W}$ be the category of well-ordered sets where morphisms are either strictly monotone maps or constant maps. The $\mathcal{W}$ is an accessible category with directed colimits but without an EM-functor [Beke, JR]. However, $\mathcal{W}$ is LS-accessible.

Let $\mathcal{K}$ be an accessible category with directed colimits. A functor $E : \textbf{\textit{Lin}} \to \mathcal{K}$ preserving directed colimits induces a geometric morphism $S(E) : \textbf{\textit{Set}}^{\Delta} \to S(\mathcal{K})$.

Following Diaconescu theorem, $PS(\textbf{\textit{Lin}}) \simeq \textbf{\textit{Lin}}$. A localic geometric morphism $G : \textbf{\textit{Set}}^{\Delta} \to S(\mathcal{K})$ yields a faithful $G(E) : \textbf{\textit{Lin}} \to PS(\mathcal{K})$ (Di Liberti 2020).