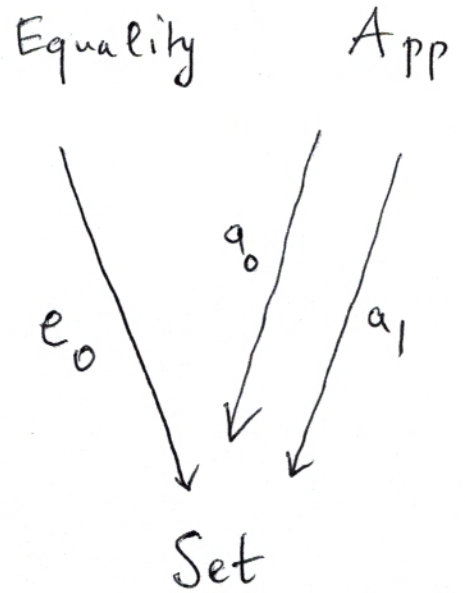
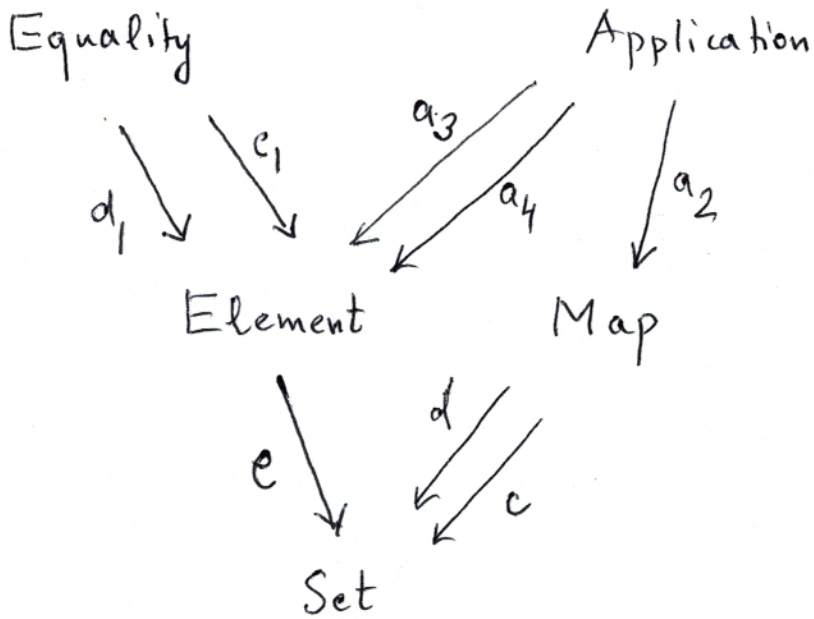


Abstract sets

Labsset

Signature $\boxed{L_{absset}}$:



$$e_0 = ed_1 = ec_1$$

$$a_0 = ea_3 = da_2$$

$$a_1 = ea_4 = ca_2$$

A concrete category (c-category) is a pair $\underline{A} = (A, a)$ where 'a' is a faithful functor $a: A \rightarrow \text{Set}$. \underline{A} is a complete c-category if 'a' is a discrete iso fibration. Equivalence for c-cats is more straightforward than for general c-cats.

$(\underline{A}_b, \text{underlying}: \underline{A}_b \rightarrow \text{Set})$ is complete, for instance.

Every c-cat can be 'completed'; the underlying category of the completion is equivalent to the original.

Let $\underline{G} = (G, \epsilon_{\underline{G}})$ be a Set-valued model of (a fragment of) ZFC. For instance, $G = L =$

the constructible sets - but if there is an inaccessible cardinal, there is a countable model of ZFC. \underline{G} gives rise to the c-cat

$$\underline{A} = \boxed{\underline{A}[\underline{G}]} = (A, a) \text{ where}$$

$$\text{Ob}(A) = G$$

$$\text{Arr}(A) = \{ (x, y, f) : x, y, "f: x \rightarrow y" \text{ in the sense of } \underline{G} \}$$

for $x \in G$, $a(x) = \{y \in G \mid y \in \underline{G}x\}$ $\overset{\text{L}_{\text{absset}}}{\square}$ \square 3
 (etc)

Re-coding c-cut $\underline{A} = (A, a)$ into

$\rightsquigarrow \boxed{M(\underline{A})} : \text{L}_{\text{absset}} \rightarrow \text{Set SET?}$

$M = M(\underline{A})$::

$$M(\text{Set}) = \text{Ob}(A)$$

$$M(\text{El}) = \{(U, u) \mid U \in \text{Ob}(A), u \in a(U)\}$$

$$M(\text{Map}) = \{(U, V, f) \mid U, V \in \text{Ob}(A); f: U \rightarrow V\}$$

$$M(\text{Eq}) = \{(U, u, v) \mid U \in \text{Ob}(A); u, v \in a(U), \textcircled{u=v}\}$$

$$M(A_{\text{pp}}) = \{(U, V, f, u, v) \mid U, V \in \text{Ob}(A); f: U \rightarrow V,$$

$$\boxed{a_0 \ a_1 \ a_2 \ a_3 \ a_4}$$

$$u \in a(U), v \in a(V)$$

$$\boxed{v = (a(f))(u)} \}$$

For the syntax, we need the five type-formation rules:

- 1 $\vdash U : \text{Set}$
- 2 $U : \text{Set} \vdash u : \text{El}(U)$
- 3 $U : \text{Set}, u, v : \text{El}(U) \vdash e : \text{Eq}(U, u, v)$
- 4 $U, V : \text{Set} \vdash f : \text{Map}(U, V)$
- 5 $U, V : \text{Set}, u : \text{El}(U), v : \text{El}(V), f : \text{Map}(U, V) \vdash a : \text{App}(U, V, u, v, f)$

Defined concepts (syntactic sugar):

examples:

Example 1:

$$X : \text{Set}, x : \text{El}(X), y \in \text{El}(X) \vdash$$

$$\boxed{x = y} \stackrel{\text{def}}{=} \exists e : \text{Eq}(X, x, y). \text{TRUE}$$

(free variables: X, x, y)

a proposition (formula)

Example 2:

Labssat 5

$X, Y: \text{Set} . x: \text{El}(X) . y \in \text{El}(Y)$

$f: \text{Map}(X, Y)$??

$f(x) = y$ $\stackrel{\text{def}}{=} \exists a: \text{App}(X, Y, f, x, y) . \text{TRUE}$

Example 3:

Context $X \xleftarrow{\pi_0} Z \xrightarrow{\pi_1} Y$ (right?)

?? Product (X, Y, Z, π_0, π_1) $\stackrel{\text{def}}{=} \equiv$

$\forall x \in X . \forall y \in Y . \exists z \in Z . \forall w \in Z$
 $\forall x: \text{El}(X)$

$[(\pi_0(w) = x \ \& \ \pi_1(w) = y) \leftrightarrow w = z]$

Example 4: axiom of existence of
binary product

ExProd $\stackrel{\text{def}}{=} \forall (X, Y) \exists (Z, \pi_0, \pi_1) \text{Product}(X, Y, Z, \pi_0, \pi_1)$

For \underline{G} model of set-theory

$L_{\text{absset}} \boxed{6}$

$$M(\underline{G}) \stackrel{\text{def}}{=} M(\underline{A}[\underline{G}]) \models \text{ExpProd}$$

is true since for $X \in \underline{G}, Y \in \underline{G}$

$$\text{we can take } Z = \underbrace{\{(x, y)_{\underline{G}} \mid x \in_{\underline{G}} X, y \in_{\underline{G}} Y\}}$$

ordered pair in the sense of \underline{G}

The universal property of the product
can be expressed and deduced from
suitable instances of an

"arrow-existence" comprehension

axiom schema in $\text{FOLDS}(L_{\text{absset}})$

which is true in $M(\underline{G})$ if $\underline{G} \models \text{ZFC}$

General 1

A global umbrella theory of structures
(such as Bourbaki's Set Theory) is
Bourbakian (new word!) if all its
definable theoretical predicates $P(A_0, A_1, \dots)$
referring to structures A_0, A_1, \dots of
respective species S_1, S_2, \dots are invariant
under the identities $\equiv_{S_1}, \equiv_{S_2}, \dots$
appropriate (adopted to be appropriate)
for the species S_1, S_2, \dots :

$$A_0 \equiv_{S_0} B_0, A_1 \equiv_{S_1} B_1, \dots P(A_0, A_1, \dots) \\ \Rightarrow P(B_0, B_1, \dots)$$

Bourbaki's own Set Theory is not
Bourbakian; Lawvere's first-order theory
of the category of sets is not Bourbakian
if "isomorphism" is the adopted notion of
identity for the usual species of structures
(groups, topological spaces, etc). Reason:

Consider the species-of-structures of General [2]
bare sets, both as S_0 and as S_1 ,

or, objects of the category Set in Lawvere's case.

The language of the umbrella theory
allows the predicate

$$P(A_0, A_1) \stackrel{\text{def}}{=} A_0 = A_1$$

since equality, $=$, is a theoretical predicate
in the first-order theory of a universe
of sets, and also, in the first-order theory of
a category. But, of course, invariance fails:

$$A_0 \cong B_0, \quad A_1 \cong B_1, \quad A_0 = A_1$$

\uparrow
isomorphism;
equipollence

$$\not\Rightarrow B_0 = B_1$$

Abstract Set Theory is Bourbakian

in the new sense since in it all

theoretical predicates are invariant

under isomorphism (intrinsic equivalence).

Internal topos theory .

General 3

becomes Bourbakian if for category theory
the FOLDS language over \mathcal{L}_{cat} is adopted

— but not without such a move.

Sanafunctors

L_{sanafun} (1)

Signature L_{sanafun} :

(L_{sanafun} is the same as L_{anafun} I used before; since the intended models now are sanafunctors = saturated anafunctors, a subclass of anafunctors in general, I changed the notation)

$L = L_{\text{sanafun}}$ contains two copies of L_{cat} , for the domain and the codomain categories of the functor being considered. L contains two additional kinds, App_0 and App_1 . These are like App in L_{absset} , for function-applications, on objects in case of App_0 , and on arrows in case App_1 . The twist is that for a functor $F: X \rightarrow A$, the application instance $(X, A, F(X) = A)$, a would-be element of App_0 , is "undesirable" because it uses equality of objects of the category A . We change this

by replacing $F(X) = A$ by $F(X) \cong A$
 and even better, by a ^{any} particular isomorphism

$i: F(X) \xrightarrow{\cong} A$. Once this is done the
 concept of sanafunctor will be readily

obtained: ~~the~~ FOLDS

a free-living sanafunctor

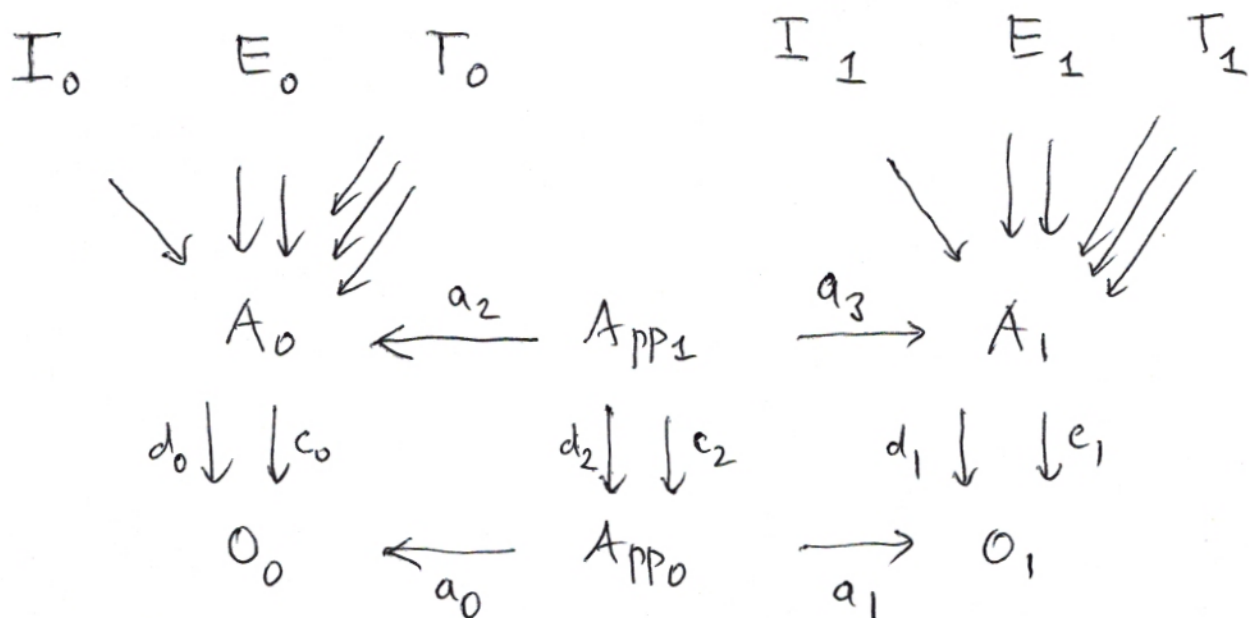
$F: X \xrightarrow{\text{sana}} A$ is a model of the

$\text{FOLDS}(L_{\text{sanafun}})$ - theory of the class of
 of all L_{sanafun} - structures $\underbrace{\text{of the form}} M(F: X \rightarrow A)$

where $M(F)$ is, of course, the L_{sanafun} -
 recoding of F . To make this reasonable and

explicit, we establish the analogs of
 the propositions stated in the treatment
 of categories in L_{cat} .

A difference to the L_{cat} case is that coming
 back from a sanafunctor to an ordinary functor
 needs an essential use of the axiom of choice

L_{sanafun}


$$d_0 a_2 = a_0 d_2$$

$$d_1 a_3 = a_1 d_2$$

$$c_0 a_2 = a_0 c_2$$

$$c_1 a_3 = a_1 c_2$$

$F: X \rightarrow A$
 functor $\longrightarrow \boxed{M(F)}; L_{\text{sanafun}} \longrightarrow \text{SET}$

as follows. The domain & codomain parts

by what happens for L_{cat} . Remains: for $M = M(F)$

$M(App_0)$, $M(App_1)$ and the related

arrows

$$\boxed{M(A_{pp_0})} = \left\{ (X, A, i) \mid \begin{array}{l} X \in \text{ob}(X), A \in \text{ob}(A) \\ i : F(X) \xrightarrow{\cong} A \end{array} \right\}$$

$$\boxed{M(A_{pp_1})} = \left\{ ((X, A, i), (Y, B, j), f : X \rightarrow Y \text{ in } X, g : A \rightarrow B \text{ in } A) \mid \right.$$

$$\begin{array}{ccc} FX & \xrightarrow[\cong]{i} & A \\ Ff \downarrow & \cong & \downarrow g \text{ commutes} \\ FY & \xrightarrow[\cong]{j} & B \end{array}$$

The action on the lower ($\dim \leq 1$) part of L_{sana} is:

