$\boxed{\text{First-order logic with dependent sorts}}$

Terminology: L a category, $K, K_p, \ldots$ objects of L
(K for 'kind')

$\boxed{\dim(K)}$ = largest $n \in \mathbb{N}$ such that

there exists:

$$K = K_n \xrightarrow[\neq 1]{f_n} K_{n-1} \xrightarrow[\neq]{f_{n-1}} \cdots \xrightarrow[\neq 1]{f_1} K_0$$

of proper (= non-identity) arrows f_i.

If no such n, $\dim(K) = \infty$.

$\boxed{L \text{ is a (FOLDS-) signature}}$ if for all $K \in \mathrm{Ob}(L)$:

1) $\dim(K) < \infty$

2) $\boxed{\tilde{K} = L(K, -)} : L \to \mathrm{Set}$ (covariant representable)

is a finite functor ($el(\tilde{K})$, the category of elements of $\tilde{K}$) has finitely many objects

Consequences: L is 1-way: $\mathrm{End}(K) = \{1_K\}$

and more generally

$$K \xrightarrow[\neq 1]{p} K_p \text{ proper} \Rightarrow \dim(K) > \dim(K_p)$$

L-structure : $M : L \to Set$ functor

(or: $M : L \to SET$)

We write $\boxed{Str(L)}$ for the functor category Set^L

Three examples to be discussed:

$\boxed{L_{cat}}$ 'cat' for 'category'

$\boxed{L_{absset}}$ 'absset' for 'abstract set'

$\boxed{L_{sanafun}}$ 'sanafun' for 'sanafunctor'
(saturated anafunctor)

Re-coding turns a classical structure into an L-structure:

$\boxed{\mathbb{C} \text{ category} \longmapsto M(\mathbb{C}) \in Str(L_{cat})}$

$\underline{A}$ concrete category (discrete iso fibration $\begin{array}{c} A \\ \downarrow a \\ Set \end{array}$)

$\longmapsto M(\underline{A}) \in Str(L_{absset})$

free-living functor $\mathbb{X} \xrightarrow{F} \mathbb{A}$

$\longmapsto M(F) \in Str(L_{sanafun})$

$\boxed{L\text{-equivalence}}$

(L, K, K', ... as before)

$\boxed{\mathring{K} : L \to \mathrm{Set}}$ is the functor, subfunctor of $\boxed{\tilde{K} = L(K, -)}$, for which $\mathrm{Ob}(\mathrm{el}(\tilde{K})) - \mathrm{Ob}(\mathrm{el}\,\mathring{K})$
$= \{(K, 1_K)\}$

$\mathring{K} \xrightarrow[i_K]{} \tilde{K}$: "sphere into ball" inclusion

Let $P, M \in \mathrm{str}(L)$, $\underline{\ell : P \to M}$ (nat. transf.)

<u>Definition</u> $\boxed{\ell \text{ is fiberwise surjective (FS)}}$

if, for all $K \in \mathrm{Ob}(L)$, ℓ has the right lifting property wrt i_K

$$\begin{tikzcd} \mathring{K} \arrow[r, "\alpha"] \arrow[d, "i_K"'] \arrow[dr, phantom, "\equiv"] & P \arrow[d, "\ell"] \\ \tilde{K} \arrow[r, "a"'] & M \end{tikzcd} \Rightarrow \exists\, \bar{a} : K \to P \quad \begin{tikzcd} \mathring{K} \arrow[r, "\alpha"] \arrow[d, "i_K"'] & P \arrow[d, "\ell"] \\ K \arrow[r, "a"'] \arrow[ur, "\bar{a}"] & M \end{tikzcd}$$

$\ell\alpha = a i_K$ $\qquad\qquad$ $\alpha = \bar{a} i_K$, $a = \underline{\ell}\bar{a}$

I'd say "trivial fibration" if there were "fibrations".
(fibrations may come later)

As a consequence, if ℓ is FS it has the RLP wrt to all monomorphisms in $\mathrm{Str}(L)$:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, tail, "i"', "\text{mono}"] \arrow[dr, phantom, "\shortparallel"] & P \arrow[d, "\ell"] \\ Y \arrow[r, "a"'] & M \end{tikzcd} \quad \Rightarrow \quad \exists\, \bar{a} : \quad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "i"'] & P \arrow[d, "\ell"] \\ Y \arrow[r, "a"'] \arrow[ur, "\bar{a}"] & M \end{tikzcd}$$

This is because the class of monomorphisms is the Gabriel-Zisman saturation of the sphere-inclusions.

Let $M, N \in \mathrm{Str}(L)$:

$\boxed{\text{An } L\text{-}\underline{\text{equivalence}}}$ $\underline{P}$ of M and N

$$\boxed{\underline{P} : M \simeq_L N}$$

is: $\underline{P} = (P, \ell, r)$:

$$\begin{tikzcd} & P \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ M & & N \end{tikzcd}$$

with both ℓ and r being FS.

$$M \simeq_L N \underset{\text{def}}{\Longleftrightarrow} \exists \underline{P} : \underline{P} : M \simeq_L N$$

More generally: let $X \in \mathrm{Set}^L$, and

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

Usually, X will be finite. X is a context/system of typed variables, α is an evaluation of X in M, similarly for β. We also say: α is an X-element of M.

Write: $\boxed{\underline{P} : (M, \alpha) \simeq_L (N, \beta)}$ if

$$\underline{P} : M \simeq_L N$$

$\underline{\text{and}}$ (next page)

... and there exists $\gamma : X \to P$ s.t.

$$\begin{tikzcd} & P \arrow[dl, "l"'] \arrow[dr, "r"] & \\ M \arrow[r, phantom, "\equiv"] & {} \arrow[r, phantom, "\equiv"] & N \\ & X \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, "\gamma"] & \end{tikzcd}$$

$$\alpha = l\gamma$$
$$\beta = r\gamma$$

(M, α), (N, β) : "augmented structures"

<u>Fact</u>: $\simeq_L$ is an equivalence relation on augmented structures (plain structures included: $X = \emptyset$)

<u>Equivalence transfer</u>:

Suppose $M \xleftarrow{\alpha} X \xrightarrow{\beta} N$

$(M, \alpha) \simeq_L (N, \beta)$

$X \xrightarrow{i} Y$ <u>monomorphism</u>

$Y \xrightarrow{\bar{\alpha}} M$ extending α : $\alpha = \bar{\alpha} i$

Then: there is

$Y \xrightarrow{\bar{\beta}} N$ extending β :: $\beta = \bar{\beta} i$

such that $(M, \bar{\alpha}) \simeq_L (N, \bar{\beta})$.

because: we have

$$(P, \ell, r) : (M, \alpha) \simeq_L (N, \beta)$$

and

$$X \xrightarrow{\gamma} P \quad \text{such that}$$

$$\begin{tikzcd}
 & P \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\
M \arrow[r, phantom, "\equiv"] & X \arrow[u, "\gamma"] \arrow[r, phantom, "\equiv"] \arrow[ul, "\alpha"] \arrow[ur] & N
\end{tikzcd}$$

is commutative

$$\begin{tikzcd}
X \arrow[r, "\gamma"] \arrow[d, "i"'] \arrow[dr, phantom, "\equiv"] & P \arrow[d, "\ell"] \\
Y \arrow[r, "\bar{\alpha}"'] & M
\end{tikzcd}$$

commutes since $\ell\gamma = \alpha = \bar{\alpha} i$

$\therefore$ since i is a mono, there is a diagonal $\bar{\gamma} : Y \to P$ such that $\gamma = \bar{\gamma} i$ & $\bar{\alpha} = \ell \bar{\gamma}$.

$$\begin{tikzcd}
 & P \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\
M & Y \arrow[l, "\bar{\alpha}"'] \arrow[u, "\bar{\gamma}"] & N \\
 & X \arrow[ul, "\alpha"] \arrow[u] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

commutes: let $\bar{\beta} =_{def} r\bar{\gamma}$. $\therefore$

$\bar{\gamma} i = r\gamma = \beta$

$$\begin{tikzcd}
 & P \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\
M & Y \arrow[l, "\bar{\alpha}"'] \arrow[u, "\bar{\gamma}"] \arrow[r, "\bar{\beta}"] & N \\
 & X \arrow[ul, "\alpha"] \arrow[u, "i"'] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

is commutative. $\square$

Equivalence transfer is used to show the $\boxed{\text{soundness of } L\text{-equivalence}}$ wrt FOLDS properties of (augmented) structures: for a FOLDS formula $\varphi(X)$,

$$(M,\alpha) \simeq_L (N,\beta) \ \& \ M \models \varphi[\alpha/X]$$
$$\Rightarrow N \models \varphi[\beta/X]$$

(with $i: X \to Y$, think of the quantifiers $\exists_i$, $\forall_i$; see later!)

$\boxed{\text{Intrinsic } L\text{-equivalence}}$

L: FOLDS signature, M: L-structure

The self-equivalence $\underline{P} = (P, l, r): M \simeq_L M$ of M extends the identity if there is $m: M \to P$ such that $lm = rm = 1_M$:

$$\begin{tikzcd}
 & P \arrow[dl, "l"'] \arrow[dr, "r"] & \\
M \arrow[r, phantom, "\equiv"] & {} \arrow[r, phantom, "\equiv"] & M \\
 & M \arrow[ul, "1_M"] \arrow[uu, "m"'] \arrow[ur, "1_M"'] &
\end{tikzcd}$$

The X_1-elements α, β of M:

$$X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$$

are <u>intrinsically (L-)equivalent</u> if

there exists $\underline{P} : (M, \alpha) \simeq_L (M, \beta)$ extending

the identity.

Notation: $\underline{P} : \alpha \underset{\text{int}}{\simeq} \beta$, $\alpha \underset{\text{int}}{\simeq} \beta$.

In the 'usual' cases, intrinsic equivalence

is the "expected" relation.

For C a category, x, y objects of C

$$x \underset{\text{int}}{\simeq} y \quad \text{in} \quad M(C) \in \mathrm{Str}(L_{cat})$$

iff $x \cong y$ (isomorphism) in C.

More generally, if X is a graph, or even

a 'category sketch' (with some triangles in the graph

marked commutative, some arrows as identities)

then $X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} C$ are intrinsically equivalent in $M(C)$

(a structure over L_{cat}) iff they are isomorphic as objects of the diagram category $\mathcal{C}^X$.

For any signature L, $X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$

and $\varphi(X)$ a FOLDS(L)-formula

Then: $\alpha \simeq_{\text{int}} \beta \;\&\; M \models \varphi[\alpha/X] \Rightarrow M \models \varphi[\beta/X]$.

as a special case of the soundness of L-equivalence.

In the case of $L = L_{\text{abset}}$ and $M \in \text{Str}(L)$

being a model of the

"minimal theory of abstract sets", $\Sigma_{\min}$

(essentially, $M \simeq_L M(\underline{A})$ for a concrete

category $\underline{A} = (A, a : A \to \text{Set})$)

$\alpha \simeq_{\text{int}} \beta$ is the same as isomorphism:

$$\alpha \cdot \cong \beta$$

Thus: $\alpha \cong \beta \;\&\; M \models \varphi[\alpha/X] \Rightarrow M \models \varphi[\beta/X]$

"Abstract set theory is Bourbakian"