

# First-order logic with dependent sorts

FOLDS (I)

Terminology:  $L$  a category,  $K, K_p, \dots$  objects of  $L$   
( $K$  for 'kind')

$\dim(K)$  = largest  $n \in \mathbb{N}$  such that

there exists:

$$K = K_n \xrightarrow[\neq 1]{f_n} K_{n-1} \xrightarrow[\neq]{f_{n-1}} \dots \xrightarrow[\neq 1]{f_1} K_0$$

of proper (= non-identity) arrows  $f_i$ .

If no such  $n$ ,  $\dim(K) = \infty$ .

$L$  is a (FOLDS-) signature if for all  $K \in \text{Ob}(L)$ :

1)  $\dim(K) < \infty$

2)  $\tilde{K} = L(K, -) : L \rightarrow \text{Set}$  (covariant representable)

is a finite functor ( $\text{el}(\tilde{K})$ , the category of elements of  $\tilde{K}$ ) has finitely many objects

Consequences:  $L$  is 1-way:  $\text{End}(K) = \{1_K\}$

and more generally

$$K \xrightarrow[\neq 1]{p} K_p \text{ proper} \Rightarrow \dim(K) > \dim(K_p)$$

$L$ -structure :  $M: L \rightarrow \text{Set}$  functor

(or:  $M: L \rightarrow \text{SET}$ )

We write  $\text{Str}(L)$  for the functor category  $\text{Set}^L$

Three examples to be discussed:

$L_{\text{cat}}$  'cat' for 'category'

$L_{\text{absset}}$  'absset' for 'abstract set'

$L_{\text{sanafun}}$  'sanafun' for 'sana-functor'  
(saturated ana-functor)

Re-coding turns a classical structure into an  $L$ -structure:

$\underline{C}$  category  $\longmapsto M(\underline{C}) \in \text{Str}(L_{\text{cat}})$

$\underline{A}$  concrete category (discrete iso fibration  $\begin{matrix} A \\ \downarrow a \\ \text{Set} \end{matrix}$ )  $\longmapsto M(\underline{A}) \in \text{Str}(L_{\text{absset}})$

free-living functor  $X \xrightarrow{F} A$   $\longmapsto M(F) \in \text{Str}(L_{\text{sanafun}})$

L - equivalence

Equ II

(L, K, K', ... as before)

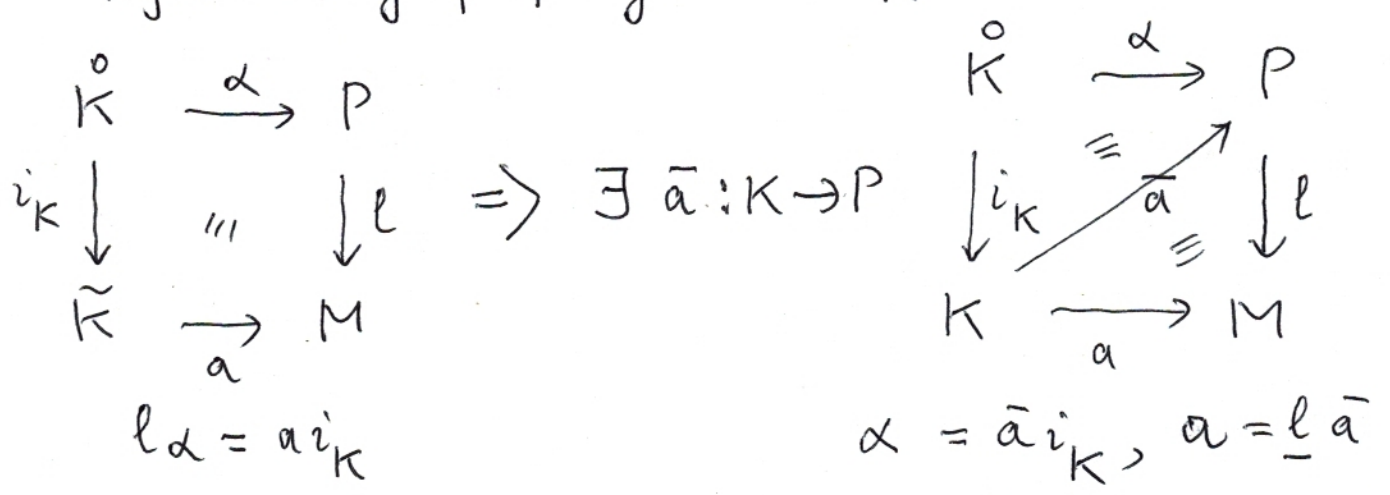
$K^{\circ} : L \rightarrow \text{Set}$  is the functor, subfunctor of  $\tilde{K} = L(K, -)$ , for which  $\text{ob}(el(K^{\circ})) = \text{ob}(el(\tilde{K})) = \{(K, 1_K)\}$

$K^{\circ} \xrightarrow{i_K} \tilde{K}$  : "sphere into ball" inclusion

Let  $P, M \in \text{str}(L)$ ,  $l : P \rightarrow M$  (nat. transf.)

Definition l is fiberwise surjective (FS)

if, for all  $K \in \text{ob}(L)$ , l has the right lifting property wrt  $i_K$



I'd say "trivial fibration" if there were "fibrations",  
 (fibrations may come later)

EQU 1.1

As a consequence, if  $l$  is FS  
 it has the RLP wrt to all monomorphisms  
 in  $\text{Str}(L)$ :

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & P \\
 \downarrow i \text{ mono} & \llcorner & \downarrow l \\
 Y & \xrightarrow{a} & M
 \end{array}
 \Rightarrow \exists \bar{a} : \begin{array}{ccc}
 X & \xrightarrow{\alpha} & P \\
 \downarrow i & \swarrow \bar{a} & \downarrow l \\
 Y & \xrightarrow{a} & M
 \end{array}$$

This is because the class of monomorphisms is  
 the Gabriel-Zisman saturation of the sphere-inclusions.

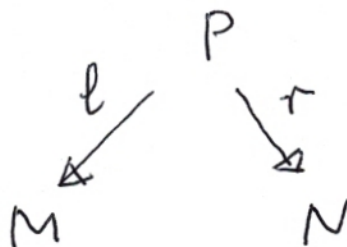
Let  $M, N \in \text{Str}(L)$ :

Equ [2]

An  $L$ -equivalence  $\underline{P}$  of  $M$  and  $N$

$$\underline{P} : M \simeq_L N$$

$\underline{P} = (P, \ell, \tau)$



with both  $\ell$  and  $\tau$  being FS.

$$M \simeq_L N \stackrel{\text{def}}{\Leftrightarrow} \exists \underline{P} : \underline{P} : M \simeq_L N$$

More generally: let  $X \in \text{Set}^L$ , and

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

Usually,  $X$  will be finite.  $X$  is a context/system of typed variables,  $\alpha$  is an evaluation of  $X$  in  $M$ , similarly for  $\beta$ . We also say:  $\alpha$  is an  $X$ -element of  $M$ .

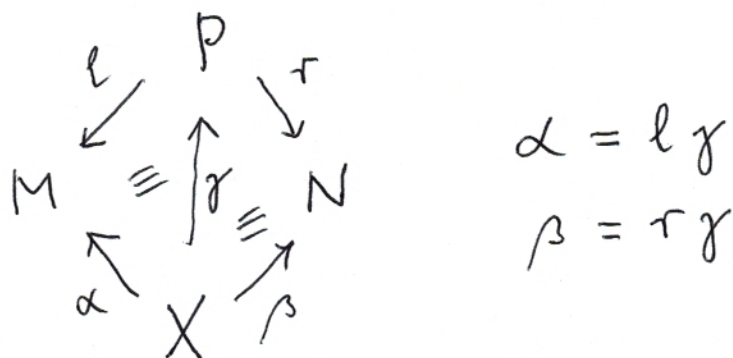
Write:  $\underline{P} : (M, \alpha) \simeq_L (N, \beta)$  if

$$\underline{P} : M \simeq_L N$$

and (next page)



... and there exists  $\gamma: X \rightarrow P$  s.t.



$(M, \alpha), (N, \beta)$  = "augmented structures"

Fact:  $\cong_L$  is an equivalence relation on augmented structures (plain structures included:  $X = \emptyset$ )

Equivalence transfer:

Suppose  $M \xleftarrow{\alpha} X \xrightarrow{\beta} N$

$(M, \alpha) \cong_L (N, \beta)$

$X \xrightarrow{i} Y$  monomorphism

$Y \xrightarrow{\bar{\alpha}} M$  extending  $\alpha$  :  $\alpha = \bar{\alpha}i$

Then: there is

$Y \xrightarrow{\bar{\beta}} N$  extending  $\beta$  :  $\beta = \bar{\beta}i$

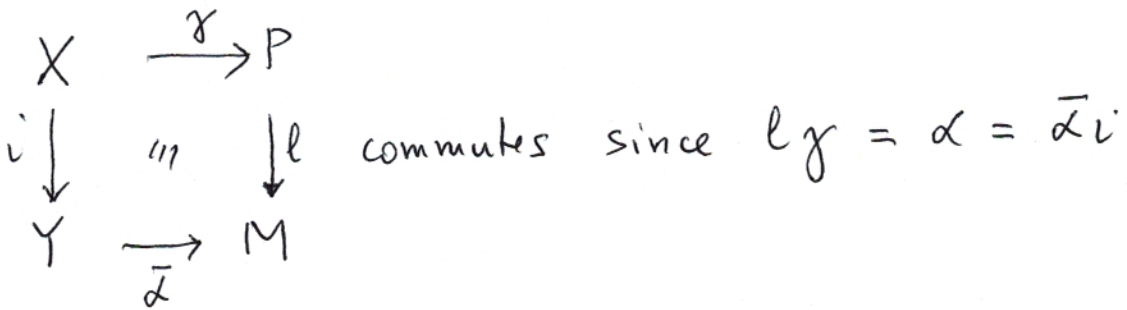
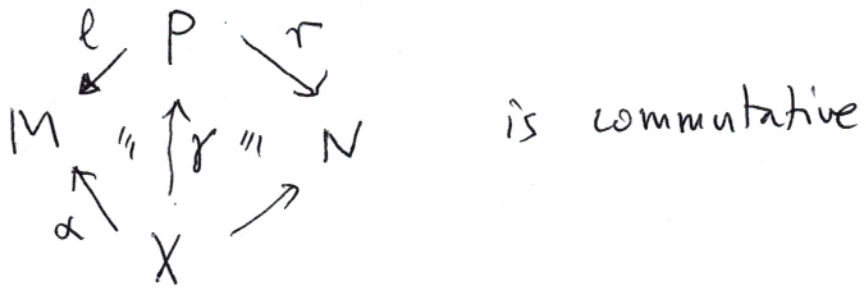
such that  $(M, \bar{\alpha}) \cong_L (N, \bar{\beta})$ .

because: we have

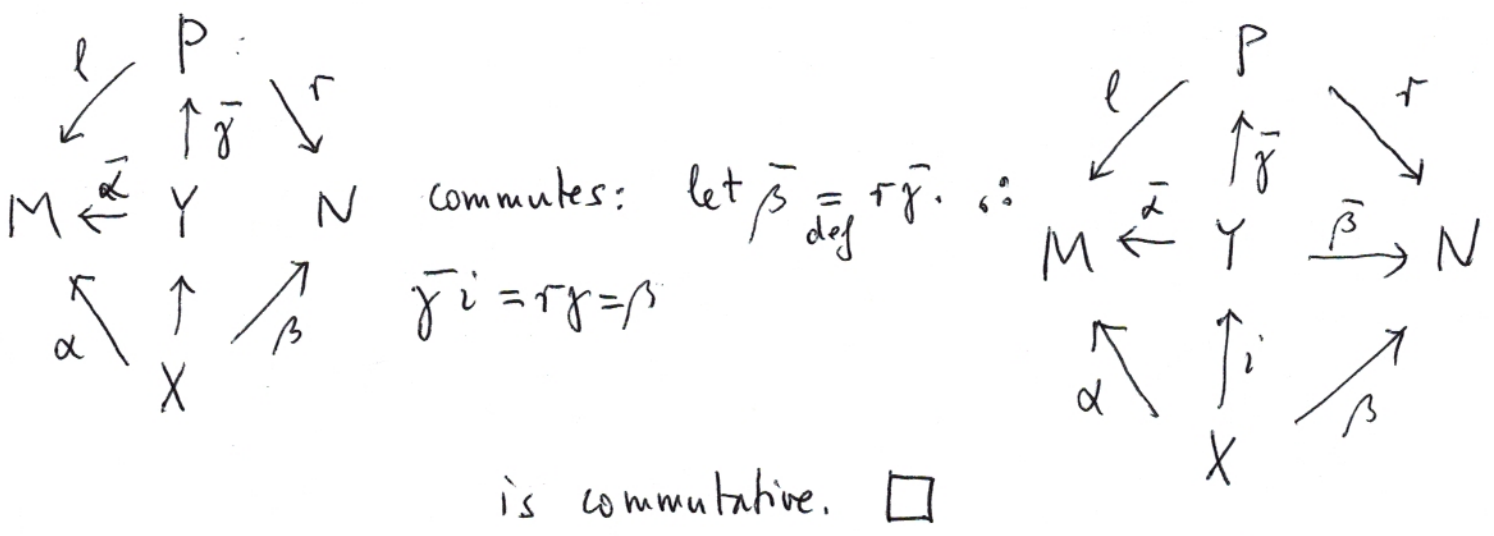
$$(P, l, r) : (M, \alpha) \cong_L (N, \beta)$$

and

$$X \xrightarrow{\gamma} P \text{ such that}$$



$\therefore$  since  $i$  is a mono, there is a diagonal  $\bar{\gamma} : Y \rightarrow P$  such that  $\gamma = \bar{\gamma}i$  &  $\bar{\alpha} = l\bar{\gamma}$ .



Equivalence transfer is used

to show the soundness of L-equivalence

wrt FOLDS properties of (augmented) structures: for a FOLDS formula  $\varphi(X)$ ,

$$(M, \alpha) \cong_L (N, \beta) \ \& \ M \models \varphi[\alpha/X]$$

$$\Rightarrow N \models \varphi[\beta/X]$$

(with  $i: X \rightarrow Y$ , think of the quantifiers

$\exists_i, \forall_i$ ; see later!)

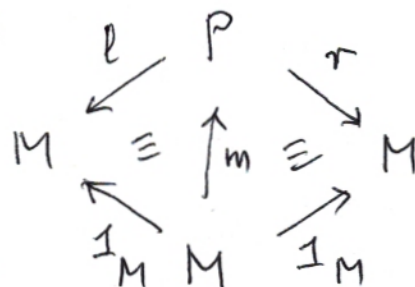
Intrinsic L-equivalence

$L$ : FOLDS signature,  $M$ : L-structure

The self-equivalence  $\underline{P} = (P, \ell, r): M \cong_L M$  of  $M$

extends the identity if there is  $m: M \rightarrow P$

such that  $\ell m = r m = 1_M$ :





The  $X$ -elements  $\alpha, \beta$  of  $M$ :

EQU [6]

$$X \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} M$$

are intrinsically (L-)equivalent if

there exists  $\underline{P} : (M, \alpha) \cong_L (M, \beta)$  extending the identity.

Notation:  $\underline{P} : \alpha \underset{\text{int}}{\cong} \beta$ ,  $\alpha \underset{\text{int}}{\cong} \beta$ .

In the 'usual' cases, intrinsic equivalence is the "expected" relation.

For  $C$  a category,  $x, y$  objects of  $C$

$$x \underset{\text{int}}{\cong} y \text{ in } M(C) \in \text{Str}(L_{\text{cat}})$$

**iff**  $x \cong y$  (isomorphism) in  $C$ .

More generally, if  $X$  is a graph, or even a 'category sketch' (with some triangles in the graph marked commutative, some arrows as identities)

then  $X \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} C$  are intrinsically equivalent in  $M(C)$

(a structure over  $L_{\text{cat}}$ ) iff they are isomorphic

as objects of the diagram category  $\mathcal{C}^X$ .

For any signature  $L$ ,  $X \xrightarrow{\alpha} M$   
 $\xrightarrow{\beta}$

and  $\varphi(X)$  a FOLDS( $L$ )-formula

Then:  $\alpha \underset{\text{int}}{\cong} \beta$  &  $M \models \varphi[\alpha/X] \Rightarrow M \models \varphi[\beta/X]$ .

as a special case of the soundness of  $L$ -equivalence.

In the case of  $L = L_{\text{absset}}$  and  $M \in \text{Str}(L)$

being a model of the

"minimal theory of abstract sets",  $\Sigma_{\text{min}}$

(essentially,  $M \underset{L}{\cong} M(\underline{A})$  for a concrete

category  $\underline{A} = (A, a: A \rightarrow \text{Set})$ )

$\alpha \underset{\text{int}}{\cong} \beta$  is the same as isomorphism:

$$\alpha \cong \beta$$

Thus:  $\alpha \cong \beta$  &  $M \models \varphi[\alpha/X] \Rightarrow M \models \varphi[\beta/X]$

"Abstract set theory is Bourbakian"