The Lindström context for FOLDS

The Lindström context for FOLDS is obtained from the "classical" Lindström context for ordinary first-order logic by replacing "isomorphism" by "FOLDS equivalence". The possibility of doing this is the first main fact. The new context is used, in the first place, to establish the soundness of the notion of FOLDS-equivalence, which is the fact that properties expressed in the language of FOLDS are invariant under FOLDS equivalence. (FOLDS equivalence is weaker than isomorphism; invariance under FOLDS-equivalence is a stronger property than invariance under isomorphism.) Secondly, the new context serves as the framework for a natural generalization for FOLDS of Lindström's classical theorem [Ordinary multi-sorted logic may be regarded as FOLDS in which all signatures have only objects ("kinds") of dimension zero or 1] Lindström's theorem is a characterization of first-order logic among all model-theoretical logics that respect isomorphism invariance. Replace "isomorphism" by "FOLDS equivalence", and you obtain the Lindström theorem for FOLDS. Not only the statement, but also the proof of the FOLDS version is a natural generalization of the original one(s).

Terminology: $L, L_0, \ldots$ denote (FOLDS-) signatures. With $X: L \to \mathbf{Set}$, the pair (L, X) is called an augmented signature. In what follows, including the Lindström theorem, we restrict attention to countable (possibly finite) L and finite X.

An (L, X)-structure (an augmented structure) is a pair (M, α) where $M: L \to \mathbf{Set}$ and $\alpha: X \to M$:

$$\begin{tikzcd} L \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "M"'{name=D}] & \mathbf{Set} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

An (L, X)- class Φ is a class of (L, X)-structures. Φ is invariant if it is closed under L-equivalence:

$$\boxed{(M,\alpha) \underset{L}{\simeq} (N,\beta) \ \& \ (M,\alpha)\in\Phi \ \Rightarrow \ (N,\beta)\in\Phi.}$$

Let $\varphi(\vec{x})$ be a FOLDS formula over L with free variables $\vec{x}$. $\vec{x}$ is the set of objects of the category of elements, $\mathrm{el}(X)$, for a finite functor $X: L\to \mathrm{Set}$. Let Φ be $\mathrm{Mod}(\varphi(\vec{x}))$, the class of all (L,X)-structures (M,α) such that $M \models \varphi[\alpha/\vec{x}]$:

in words: M satisfies φ when each $x\in\vec{x}=\mathrm{el}(X)$ is interpreted by $\alpha(x)\in M$ (more precisely, for $x\in X(K)$, by $\alpha_K(x)\in M(K)$; here $K\in \mathrm{Ob}(L)$).
The soundness of FOLDS equivalence is the statement that Φ, obtained from any $\varphi(\vec{x})$, is an invariant class.

We organize all invariant classes into a simple-minded super-large (I believe in Grothendieck universes!) discrete opfibration $\mathcal{L}$ over a category $\mathcal{L}_0$ of augmented signatures.

$\mathcal{L}$ is a Lawerean hyperdoctrine, with familiar logical operations (even infinitary ones, and "second-order" ones) given as adjoints.

I am not going to spell out the full structure of $\mathcal{L}$, only the part that is relevant to finitary first-order logic with dependent sorts. In fact, the full structure is revealed only when the base category $\mathcal{L}_0$ is enlarged with new arrows.

$\mathcal{L}_0$: the category of augmented signatures

For L_0, L FOLDS signatures, a functor $F: L_0 \to L$ is initial if F is injective on objects, F is full and faithful and "L_0 is closed downward in L":

$$\boxed{\begin{array}{l} U \in Ob(L_0),\ K \in Ob(L),\ f: K \longrightarrow F(U) \text{ in } L \\ \qquad\qquad \Rightarrow \exists V \in Ob(L_0).\ K = F(V). \end{array}}$$

Example: L_{cat} has two obvious initial embeddings in(to) L_{anafun}:

$$\begin{tikzcd} L_{cat} \arrow[r, shift left, "left"] \arrow[r, shift right, "right"'] & L_{anafun} \end{tikzcd}$$

Objects of $\mathcal{L}_0$: augmented signatures (L, X).

Morphisms of $\mathcal{L}_D$:

$$\boxed{\dfrac{(L_0, X_0) \xrightarrow{(F,\xi)} (L, X)}{\begin{tikzcd} L_0 \arrow[rr, "F"] \arrow[dr, "X_0"'] & \arrow[d, phantom, "\stackrel{\xi}{\Rightarrow}"] & L \arrow[dl, "X"] \\ & \mathrm{Set} & \end{tikzcd} \qquad \xi : X_0 \longrightarrow XF}}$$

(ξ is "substitution of variables").

There is an obvious associative and unitary composition:

$$\boxed{\begin{tikzcd} (L_1, X_1) \arrow[r, "(F_1,\xi_1)"] \arrow[rr, bend right, "(F_2F_1,\,(\xi_2F_1)\xi_1)"'] & (L_2, X_2) \arrow[r, "(F_2,\xi_2)"] & (L_3, X_3) \end{tikzcd}}$$

Let me use the symbols $\underline{L}$, $\underline{L}_0$, ... for augmented signatures: $\underline{L} = (L, X)$, $\underline{L}_0 = (L_0, X_1)$, etc.

Let $\mathcal{L}(\underline{L})$ denote the collection of all invariant $\underline{L}$-classes: $\Phi \in \mathcal{L}(\underline{L}) \Leftrightarrow \Phi$ is an invariant $\underline{L}$-class.

To make the assignment

$$\underline{L} \longmapsto \mathcal{L}(\underline{L})$$

into a functor

$$\mathcal{L} : \mathcal{L}_0 \longrightarrow \mathrm{SET}$$

we need the conepts of <u>reduct</u> and co-<u>reduct</u>.

For $\underline{M} = (M, \alpha) \in \mathcal{L}(\underline{L})$

$$\boxed{\begin{tikzcd} L \arrow[r, shift left=2, "X"] \arrow[r, shift right=2, "M"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Set} \end{tikzcd}}$$

and $\boxed{\underline{F} = (F, \xi) : \underline{L}_0 \to \underline{L}}$,

$$\boxed{\begin{tikzcd} L_0 \arrow[r, shift left=2, "X_0"] \arrow[r, shift right=2, "MF"'] \arrow[r, phantom, "\Downarrow\xi"] & \mathrm{Set} \end{tikzcd}}$$

the <u>reduct</u> of $\underline{M}$, denoted $\underline{M}\restriction\underline{F}$, is the augmented structure

$$\underline{M}\restriction\underline{F} = (M\restriction F, \alpha\restriction\underline{F}),$$

where $M\restriction F \underset{\mathrm{def}}{=} MF : L_0 \to \mathrm{Set}$

and $\alpha\restriction\underline{F} \underset{\mathrm{def}}{=} (\alpha F)\xi : X_0 \to MF$

(compare:

$$\boxed{\begin{tikzcd} L_0 \arrow[r, "F"'] \arrow[rr, bend left=50, "X_0"] \arrow[rr, phantom, shift left=4, "\Downarrow\xi"] & L \arrow[r, shift left=2, "X"] \arrow[r, shift right=2, "M"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Set} \end{tikzcd}}$$

)

For $\underline{F} : \underline{L}_0 \to \underline{L}$ and $\Phi \in \mathcal{L}(\underline{L}_0)$,

the co-reduct along $\underline{F}$ of Φ, $\boxed{\Phi \upharpoonleft \underline{F}}$,

is defined by:

for $\underline{M} \in \mathrm{Str}(\underline{L})$,

$$\boxed{\underline{M} \in \Phi \upharpoonleft \underline{F} \underset{\text{def}}{\iff} \underline{M} \upharpoonright \underline{F} \in \Phi}.$$

It is an elementary but, of course, essential observation, senstively dependent on the initiality of $F : L_0 \to L$ in $\underline{F} = (F, \xi)$, that co-reduction preserves invariance:

$$\boxed{\Phi \in \mathcal{L}(\underline{L}_0) \Rightarrow \Phi \upharpoonleft \underline{F} \in \mathcal{L}(\underline{L})}.$$

We have the functor $\mathcal{L} : \mathcal{L}_0 \longrightarrow \mathrm{SET}$:

$$\boxed{\begin{array}{ll} \underline{L} & \longmapsto \mathcal{L}(\underline{L}) \\ \underline{L}_0 \xrightarrow{\underline{F}} \underline{L} & \longmapsto \\ & \left\{ \begin{array}{l} \mathcal{L}(\underline{L}_0) \longrightarrow \mathcal{L}(\underline{L}) \\ \quad \Phi \longmapsto \Phi \upharpoonleft \underline{F} \end{array} \right. \end{array}}$$

$\boxed{\text{The logical operations:}}$ $\boxed{T_{\underline{L}},\ \wedge^{I}_{\underline{L}},\ \vee^{I}_{\underline{L}},\ \neg_{\underline{L}},\ \forall_{\underline{F}},\ \exists_{\underline{F}}}$:

- given $\underline{L} \in \mathcal{L}_0$, the constant (0-ary operation)

$$\boxed{T_{\underline{L}} \in \mathcal{L}(\underline{L})}$$

where $T_{\underline{L}}$ = class of all $\underline{L}$-structures.

- for given $\underline{L}$, finite set I: the "Booleans":

$$\boxed{\wedge^{I}_{\underline{L}} : \mathcal{L}(\underline{L})^{I} \longrightarrow \mathcal{L}(\underline{L})}$$

$$\boxed{\langle \Phi_i \rangle_{i\in I} \longmapsto \bigcap_{i\in I} \Phi_i} \quad \text{(intersection)}$$

$$\boxed{\vee^{I}_{\underline{L}} : \mathcal{L}(\underline{L})^{I} \longrightarrow \mathcal{L}(\underline{L})}$$

$$\boxed{\langle \Phi_i \rangle_{i\in I} \longmapsto \bigcup_{i\in I} \Phi_i} \quad \text{(union)}$$

$$\boxed{\neg^{I}_{\underline{L}} : \mathcal{L}(\underline{L}) \longrightarrow \mathcal{L}(\underline{L})}$$

$$\boxed{\Phi \longmapsto T_{\underline{L}} - \Phi} \quad \text{(complement)}$$

(1) For $\underline{F} : \underline{L}_0 \to \underline{L}$, we say that $\underline{F}$ is (first-order) quantifiable if

$\underline{L}_0 = (L, X_0)$, $\underline{L} = (L, X)$

with the same L; $L_0 = L$; and

$\underline{F} = (F = \mathrm{Id}_L, \xi)$, $\xi : X_0 \to X$

and ξ is a $\boxed{\text{monomorphism}}$ in $\mathbf{Set}^L$.

For $\underline{F} : \underline{L}_0 \to \underline{L}$, quantifiable, we have the operations, the quantifiers:

$$\boxed{\begin{array}{l} \forall_{\underline{F}} : \mathcal{L}(\underline{L}) \longrightarrow \mathcal{L}(\underline{L}_0) \\ \exists_{\underline{F}} : \mathcal{L}(\underline{L}) \longrightarrow \mathcal{L}(\underline{L}_0) \end{array}}$$

satisfying the adjunction properties:

for all $\Psi \in \mathcal{L}(\underline{L}_0)$ and $\Phi \in \mathcal{L}(L)$:

$$\boxed{\text{iff } \frac{\Psi \subseteq \forall_{\underline{F}}(\Phi)}{\Psi \upharpoonright \underline{F} \subseteq \Phi}} \quad \text{and} \quad \boxed{\frac{\exists_{\underline{F}}(\Phi) \subseteq \Psi}{\Phi \subseteq \Psi \upharpoonright \underline{F}} \text{ iff}}$$

The essential facts are: if $\underline{F}$ is quantifiable, and Φ is invariant, then $\forall_{\underline{F}}(\Phi)$, $\exists_{\underline{F}}(\Phi)$ exist and they are invariant. These facts are proved by explicit formulas for these classes (that are left as an exercise to find), and the $\boxed{\text{equivalence transfer}}$ argument (see EQU $\boxed{4}$).

Generalized (model-theoretical) first-order logics with dependent sorts:

A generalized FOLDS G is a subfunctor of $\mathcal{L} : \mathcal{L}_0 \to SET$ which, as an op-fibration, is closed under the logical operations. Explicitly: G is given, for each $\underline{L} \in \mathcal{L}_0$, by a subcollection $G(\underline{L})$ of $\mathcal{L}(\underline{L})$ such that each of the logical

operations: constant $\in \mathcal{L}(\underline{L})$

$$\wedge^{I}_{\underline{L}}, \vee^{I}_{\underline{L}} : \mathcal{L}(\underline{L})^{I} \longrightarrow \mathcal{L}(\underline{L})$$

$$\neg_{\underline{L}} : \mathcal{L}(\underline{L}) \longrightarrow \mathcal{L}(\underline{L})$$

and $\boxed{\exists_{\underline{F}}, \forall_{\underline{F}} : \mathcal{L}(\underline{L}_0) \longrightarrow \mathcal{L}(\underline{L}),}$

all restrict to operations "on G":

$$\boxed{\begin{array}{c} T_{\underline{L}} \in G(\underline{L}) \\ \wedge^I_{\underline{L}}, \ \vee^I_{\underline{L}} : G(\underline{L})^I \longrightarrow G(\underline{L}) \\ \neg_{\underline{L}} : G(\underline{L}) \longrightarrow G(\underline{L}) \\ \exists_{\underline{F}}, \forall_{\underline{F}} : G(\underline{L}_0) \longrightarrow G(\underline{L}) \\ \text{(here } \underline{F} \text{ is first-order quantifiable).} \end{array}}$$

The fact that G is closed under coreduction:

for (arbitrary) $\underline{F} : \underline{L}_0 \to \underline{L}$,

$$\Phi \in G(\underline{L}_0) \Rightarrow \Phi \upharpoonright \underline{F} \in G(L)$$

is the same as the condition that G is a subfunctor of $\mathcal{L}$.

Suppose we have, for every $\underline{L} \in \mathcal{L}_0$, an (arbitrary) subcollection $G_0(\underline{L})$ of $\mathcal{L}(\underline{L})$. Then there is a least generalized FOLDS G that contains G_0;

$$G_0(\underline{L}) \subseteq G(\underline{L}).$$

$\mathcal{G}$ is the $\boxed{\text{closure}}$ of $\mathcal{G}_0$ under coreduction and the logical operators. We denote this $\mathcal{G}$ by $\boxed{\mathcal{G}\langle\mathcal{G}_0\rangle}$.

There is a minimal (minimum) generalized FOLDS, $\boxed{\text{TRUE-FOLDS} = \mathcal{G}\langle\emptyset\rangle}$, the closure of the system of empty subclasses in each $\mathcal{L}(\underline{L})$. If $\underline{L}$ is finite, then $\Phi \in \mathcal{G}\langle\emptyset\rangle(\underline{L})$ iff $\Phi = \mathrm{Mod}(\varphi(\vec{x}))$ for a FOLDS $\underbrace{\text{formula}}_{\varphi(\vec{x})}$ over $\underline{L} = (L, X)$ ($\vec{x} = el(X)$ as before). For general (countable) $\underline{L}$, $\Phi \in \mathcal{G}\langle\emptyset\rangle(\underline{L})$ iff for some finite $\underline{L}_0$ and initial inclusion $\underline{F} : \underline{L}_0 \to \underline{L}$ and for some $\Phi_0 \in \mathcal{G}\langle\emptyset\rangle(\underline{L}_0)$, $\Phi = \Phi_0 \upharpoonright \underline{F}$.

In brief, $\boxed{\text{TRUE-FOLDS} = \mathcal{G}\langle\emptyset\rangle}$ is the totality of finitary first-order logic with dependent sorts interpreted in Set-valued structures.

Model-theoretic condition on a generalized FOLDS G:

$\boxed{\text{CCDLS}}$: countable compactness with the downward Löwenheim-Skolem property

For any $\underline{L} = (L, X) \in \mathcal{L}_0$,

$\Phi_n \in G(\underline{L})$ for $n \in \mathbb{N}$

such that for all $n \in \mathbb{N}$

$\bigcap_{k \leq n} \Phi_k \neq \emptyset$ (non-empty),

the total intersection $\bigcap_{n \in \mathbb{N}} \Phi_n$ is non-empty,

and in fact contains a $\boxed{\text{countable}}$ structure

$\underline{M} = (M, \alpha) \in \bigcap_{n \in \mathbb{N}} \Phi_n$.

$\underline{M}$ being countable means that for each $K \in \mathrm{Ob}(L)$, $M(K)$ is a countable set.

G = TRUE-FOLDS satisfies CCDLS (and more) as a consequence of the fact that ordinary (multisorted) first-order logic satisfies CCDLS.

Lindström theorem for FOLDS

Let $\underline{L} = (L, X)$ be a finite augmented signature (L finite category),

Φ an invariant (L, X)-class.

$\boxed{G\langle\Phi\rangle}$ denotes the generalized FOLDS generated by the singleton $\{\Phi\}$.

Assume that $G\langle\Phi\rangle$ satisfies the CCDLS condition.

Then Φ is a member of TRUE-FOLDS$(\underline{L})$, i.e. is defined by a single FOLDS formula $\varphi(\vec{x})$.

Equivalently: $G\langle\Phi\rangle$ equals the minimal generalized FOLDS, TRUE-FOLDS.

As an immediate consequence, the analogous assertion for $G\langle\Phi_0, \Phi_1, \ldots\rangle$ is also true.